OPTIMIZATION OF SYSTEMS RELIABILITY BY THE DISCRETE MAXIMUM PRINCIPLE

by

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1. INTRODUCTION

The economic feasibility of the system is determined by the failure characteristics of the system. To avoid too frequent failures of the system, it is necessary that the system operates with sufficient reliability. The system reliability may be increased by employing more reliable components which form the structure of the system. This method may be applied upto a certain level of reliability, beyond which a little increase in reliability will increase the cost tremendously. Another way to increase the systems reliability for a system composed of less reliable components is to employ redundant components. In case of parallel redundancy number of components are operating simultaneously out of which only one component is required for successful operation of the system. In case of standby redundancy, number of components are in reserve which may be switched into service in case of component failure.

The reliability of any system may be increased by employing a large number of redundant units. Usually there are restrictions on the number of units that may be employed. For example, increasing the number of elements results into increased cost, volume, weight etc. The resources such as these being limited there arises a reliability optimization problem. There are two types of problems that are usually encountered. (1) Maximization of system reliability subject to linear and/or nonlinear constraints. (2) Minimization of cost of the system while achieving a certain minimum level of reliability.

The purpose of this report is to obtain the solutions to such problems by the discrete maximum principle. The problems with single or multiple, linear or nonlinear constraints of the separable and monotonically increasing type are considered.

The following people have worked on these problems with linear constraints. Hees and Meerendonk (6) used graphs to obtain an approximate solution. The lagrange multiplier technique is used by Moskowitz and McLean (10) and Mine (9). Kettelle (7) used dynamic programming to obtain an exact solution to one constraint problem. Bellman and Dreyfus (3) solved a two constraint problem by dynamic programming using lagrange multiplier to convert the problem to one constraint problem. The work of Kettelle was extended for multiple constraints by Proschan and Bray (11). There are number of defects with their algorithm such as very few constraints can be handled, due to approximation made for the computation of reliability, the optimum solution is not guaranteed etc. In addition, the defect of the computational efficiency is mentioned in their paper.

> Even though the solution to a problem may appear to be practical in the sense that the basic quantities do not exceed the program limitations, it may turn out to be impractical because the dominating sequences developed exceed computer capacity. It is also possible that the computer time required may be excessive.

Tillman (13) using integer programming solved the problems of maximizing systems reliability subject to multiple separable constraints. One of the problems he solved is similar to Example 5 of this report.

Kettelle (7) worked with a nonlinear constraint problem of minimizing a linear cost function while achieving a certain minimum level of reliability. This problem is the same as Example 3 given in this report. Tillman (13) applied integer programming for minimization of a nonlinear cost function while achieving a minimum level of reliability subject to multiple constraints. A similar problem of maximizing system reliability subject to nonlinear constraints is solved in Example 6.

The first attempt to apply the discrete maximum principle to reliability problems was made by Fan, Wang, Tillman and Hwang (7). An objective function which is the expected profit for the system is considered.

In section 2 of this report the basic definition of reliability and review of systems reliability is presented. Two types of problems that arise in system reliability optimization are stated. In section 3, the basic algorithm of a discrete version of the maximum principle and the modification of the basic algorithm for fixed end conditions are given. A recurrence relation for a one dimensional process is derived.

Section 4 deals with single constraint problems. Maximization of systems reliability subject to weight constraint is given in Example 1. The same problem, when the maximum number of elements at any stage is restricted to some value, is solved in Example 2. Both these problems deal with linear constraints. In Example 3, a linear cost function is minimized while achieving a certain minimum level of reliability. Example 4 deals with the maximization of system reliability subject to a nonlinear constraint.

Optimization of problems involving separable functions by the discrete version of the maximum principle is discussed in section 5. The technique developed in this section is used to solve systems reliability optimization problems with multiple linear and nonlinear constraints in section 6. In Example 5, the maximization of systems reliability subject to linear, weight, cost and volume constraints is given. Example 6 deals with the same problem but nonlinear constraints are considered.

It must be noted that the number of parallel redundant unit is, in reality, positive integer, however, it is assumed that the number is continuous variable. Therefore the solutions are first obtained for continuous variables, then they are round off to integers. It is assumed that the integer solutions obtained by rounding off are still optimum. In case of nonlinear constraints, condition of convexity or concavity is assumed for insuring the global optimum. If these two assumptions are not valid, a searching method may be used, starting with the solution obtained by the methods given in this report. Zero one integer programming formulation given in section 7 may be used for round off.

2. REVIEW OF SYSTEMS RELIABILITY

Reliability is the probability that a certain device will function successfully. Bazovsky (2) defines the reliability as the probability of a device performing its purpose adequately for the period of time intended and under the operating conditions encountered. The theory of reliability is well explained in references (1) and (2).

In practical reliability systems, a number of components are connected together in some desired manner. The reliability models are concerned with the relationship between the components of a system and their effect on the performance of that system. This relationship consists of two things: first, the output of the individual component and second, the functional relationship among all components working together as a system. Here, only the result of the operation as a success or failure is considered neglecting any variation in the quality of the performance in between these two states.

The output of the system is governed by the functional relationship among different components. The system could be designed so that for its operation all components must function satisfactorily. Such a system is called a series system. The N-stage series system is shown in Fig. 1. On the other hand the system could be designed so that in the event of failure of any one of the components, another is already in the system to take over the operation; therefore, the system will fail when all the components fail and for successful operation at least one component must perform satisfactorily. Such a system configuration is called a parallel system. The N-stage parallel system is shown in Fig. 2. The property of the parallel system where out of N simultaneously operating components, only one is needed for successful operation and the remaining N-1 components serve as a reserve, is frequently

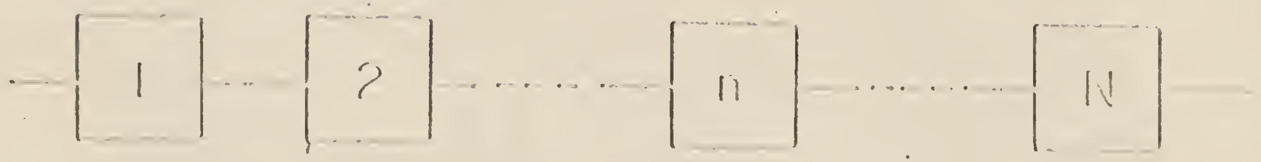

Fig. 1. A N-stage series system.

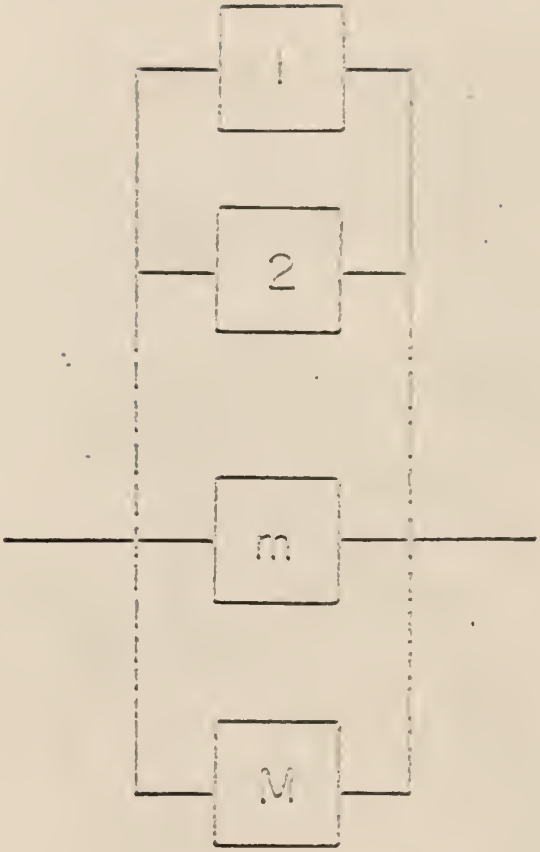

Fig. 2. A M-stage parallel system.

described by the engineering term "redundancy". Most of the practical systems include both series and parallel relationships. Such a mixed system with N subsystems in series and each subsystem with some number of redundant units in parallel is shown in Fig. 3.

The system shown in Fig. 1 consists of a number of components combined in series in which if one fails the system fails. For N components in series the system reliability is given by

$$R_s = \prod_{n=1}^{N} R^n , \qquad (2\text{-}1)^*$$

where R^n is the reliability of the nth component in series in the system.

For a system with M components in parallel as shown in Fig. 2, the unreliability, which is the probability that all M components will fail, is given by

$$\prod_{i=1}^{M} (1-R^i) .$$

Therefore, for a parallel configuration the system reliability is given by

$$R_s = 1 - \prod_{i=1}^{M} (1-R^i) . \qquad (2\text{-}2)$$

If the reliability of each component of the parallel system is equal to R, then equation (2-2) becomes

$$R_s = 1-(1-R)^M . \qquad (2\text{-}2a)$$

Now if the N stages in the mixed system shown in Fig. 3 have M^n-1 parallel redundancies at the nth stage, then the system reliability is

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[*] The superscript n indicates the stage number. The exponents are written with parentheses or brackets such as $(x^n)^2$ or $\{T^n(x^{n-1};\theta^n)\}^2$.

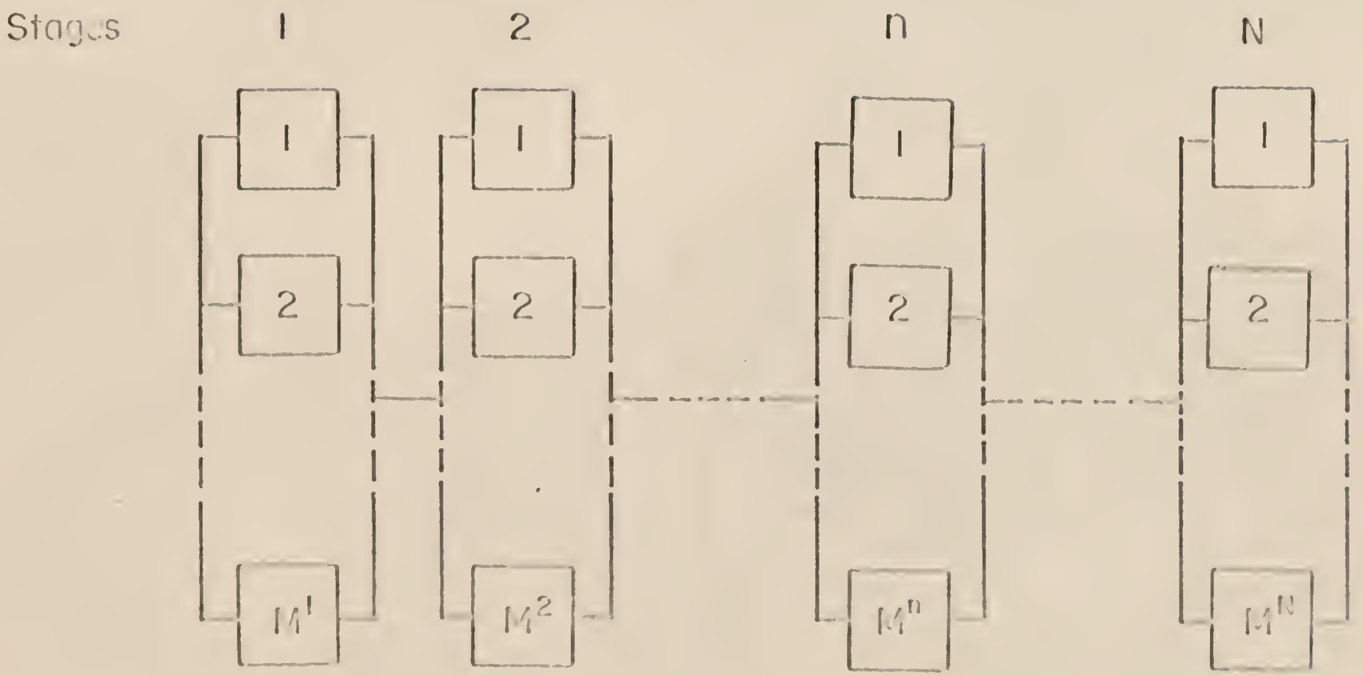

Fig. 3. A mixed system with N-stages in series where components are in parallel at each stage.

$$R_s = \prod_{n=1}^{N} 1-(1-R^n)^{M^n} . \qquad (2-3)$$

NATURE OF THE PROBLEM:

Consider the example of the three stage series system (5). The reliability of each component is $R^1 = 0.8$, $R^2 = 0.5$, and $R^3 = 0.25$. This gives the total system reliability as

$$R_s = (0.8)x(0.5)x(0.25)$$

$$= 0.1 .$$

Now if we add one redundant unit at each stage then the system reliability can be calculated from equation (2-3), which is given by

$$R_s = 0.315 .$$

Thus the reliability is increased by a factor of about 3. By adding more redundant units we can raise the level of the reliability to any desired value. Of course each time another duplicate component is installed, the total weight, cost and volume increases. Usually these resources are limited. This gives rise to different types of reliability optimization problems.

Type 1.

The problem of maximizing the reliability of an N-stage series system with redundancy in parallel subject to a single or multiple linear and/or nonlinear constraints can be stated as follows:

Maximize

$$R_s = \prod_{n=1}^{N} R^n ,$$

subject to

$$f_i(\theta^n) \le b_i , \qquad i = 1, 2, . . . , r,$$

where

R_s = the total system reliability,

N = the number of stages,

R^n = the reliability of nth stage,

θ^n = the number of elements at nth stage,

$r_i^n(\theta^n)$ = relation representing the amount of ith resource consumed at nth stage as a function of θ^n. This can be linear and/or nonlinear,

r = the number of constraints,

b_i = the total amount of ith resource available.

Often there is a restriction on the maximum number of elements that can be used at any one stage. In such a case the additional constraint can be written as

$$\theta^n \leq m,$$

where m is the maximum number of elements allowed at the stage.

Examples 1, 2, 4, 5 and 6 in this report are of the type discussed above.

Type 2.

The problem of optimizing a linear objective function while maintaining a certain minimum level of reliability can be stated as follows:

Optimize

$$S = f(\theta^n),$$

subject to

$$\prod_{n=1}^{N} R^n \geq R,$$

where

$f(\theta^n)$ = the objective function as a function of θ^n,

R = the minimum desired level of reliability.

Example 3 is similar to this type. It should be pointed out that the methods used in this report can be applied to problems of this type where the objective function is nonlinear and there are additional linear and/or nonlinear constraints on the system.

3. REVIEW OF A DISCRETE FORM OF THE MAXIMUM PRINCIPLE

A multistage system with N stages in series is shown in Fig. 4. The process consists of N stages connected in series. The state of the process stream denoted by an s-dimensional vector, $x = (x_1, x_2, \ldots, x_s)$, is transformed at each stage according to an r-dimensional decision vector, $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, which represents the decision made at that stage. The transformation equation at nth stage may be written, in vector form, as follows

$$x^n = T^n(x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \tag{3-1}$$

$$x^0 = \alpha.$$

The optimization problem associated with such a system is to find a sequence of decision variables θ^n, $n = 1, 2, \ldots, N$, subject to constraints

$$\psi_i^n(\theta_1^n, \theta_2^n, \ldots, \theta_r^n) \leq 0, \qquad n = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, r, \tag{3-2}$$

which makes a function of final state variables

$$S = \sum_{i=1}^{s} c_i x_i^N, \qquad c_i = \text{constant}, \tag{3-3}$$

an extremum when the initial condition $x^0 = \alpha$ is given.

The procedure for solving such an optimization problem by a discrete version of the maximum principle is to introduce an s dimensional adjoint vector z^n and a Hamiltonian function H^n which satisfy the following relations

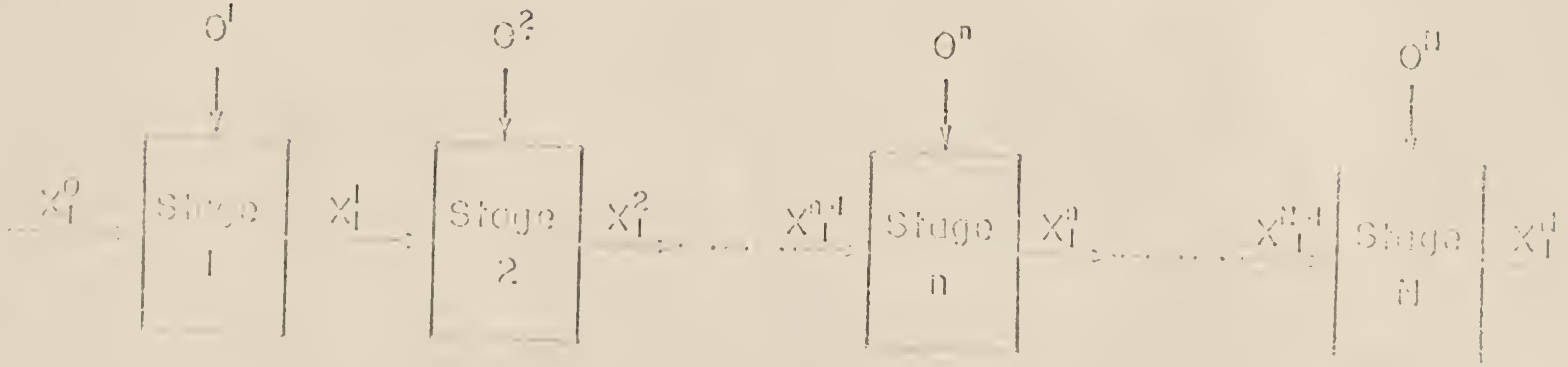

Fig 4. Multistage decision process.

$$H^n = \sum_{i=1}^{s} z_i^n T_i^n (x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \qquad (3-4)$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}}, \qquad n = 1, 2, \ldots, N; \quad i = 1, 2, \ldots, s,$$

and

$$z_i^N = c_i, \qquad i = 1, 2, \ldots, s. \qquad (3-6)$$

If the optimal decision vector function, $\bar{\theta}^n$, which makes the objective function S an optimum, is interior to the set of admissible decisions, θ^n, [the set given by equation (3-2)], a necessary condition for S to be an (local) extremum with respect to θ^n is

$$\frac{\partial H^n}{\partial \theta^n} = 0, \qquad n = 1, 2, 3, \ldots, N. \qquad (3-7)$$

If $\bar{\theta}^n$ is at a boundary of the set, it can be determined from the condition that H^n is (locally) an extremum.

This is the basic algorithm of the discrete maximum principle. In the above formulation of the problem we see that only initial conditions of the state variables are fixed, but in practice many cases might arise where the final conditions of the state variable might also be fixed. For example the final conditions x_a^N and x_b^N may be preassigned and the objective function given as

$$S = \sum_{\substack{i=1 \\ i \neq a \\ i \neq b}}^{s} c_i x_i^{\;} . \qquad (3\text{-}8)$$

Under such conditions the basic algorithm is still applicable, except that equation (3-6) is changed to

$$z_i^N = c_i , \qquad i = 1, 2, \ldots, s, \quad i \neq a, b. \qquad (3\text{-}9)$$

OPTIMIZATION OF ONE DIMENSIONAL PROCESSES (4):

If a multistage decision process can be completely characterized for the purpose of optimization by a single state variable, the process is called a one-dimensional multistage decision process.

For one dimensional process, there is only one state variable x_1, satisfying the performance equation

$$x_1^n = T^n(x_1^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N. \qquad (3\text{-}10)$$

In general, the objective function to be optimized is the sum of a certain function of x_1 and θ over all stages of the system such as

$$\sum_{n=1}^{N} G(x_1^{n-1}; \theta^n) .$$

The optimization problem associated with such a process is to find a sequence of decision variables θ^n, $n = 1, 2, \ldots, N$ so as to maximize

$$\sum_{n=1}^{N} G(x_1^{n-1}; \theta^n),$$

when x_1^0 is given. Introducing a new state variable x_2^n satisfying

$$x_2^n = x_2^{n-1} + G(x_1^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \tag{3-11}$$

$$x_2^0 = 0.$$

Therefore, we see that the objective function is given by

$$S = \sum_{n=1}^{N} G(x_1^{n-1}; \theta^n) = x_2^N . \tag{3-12}$$

Thus the problem is transformed into the standard form in which a sequence of θ^n, n = 1, 2, . . . , N is to be chosen so as to optimize x_2^N for a process described by equations (3-10) and (3-11). x_1^n is called the primary state variable and x_2^n is the secondary state variable.

Then the Hamiltonian function H^n defined by equation (3-4) can be written as

$$H^n = z_1^n \, T\,(x_1^{n-1}; \theta^n) + z_2^n \left(x_2^{n-1} + G\,(x_1^{n-1}; \theta^n)\right) . \tag{3-13}$$

According to equation (3-5), the recurrence relations for the adjoint vector elements z_1 and z_2 are found to be

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = \frac{\partial T\,(x_1^{n-1}; \theta^n)}{\partial x_1^{n-1}} \; z_1^n + \frac{\partial G\,(x_1^{n-1}; \theta^n)}{\partial x_1^{n-1}} \; z_2^n, \tag{3-14}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n , \qquad (3\text{-}15)$$

$$n = 1, 2, \ldots , N.$$

Since the objective function is

$$S = \sum_{i=1}^{2} c_i \, x_i^N = x_2^N ,$$

$$c_1 = 0, \; c_2 = 1 .$$

Thus we obtain

$$z_1^N = 0, \qquad (3\text{-}15a)$$

$$z_2^N = 1. \qquad (3\text{-}15b)$$

Combining equation (3-15b) and (3-15) and substituting in equation (3-14), gives

$$z_2^n = 1, \qquad n = 1, 2, \ldots , N, \qquad (3\text{-}16)$$

and

$$z_1^{n-1} = \frac{\partial T\,(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ z_1^n + \frac{\partial G\,(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ , \qquad (3\text{-}17)$$

$$n = 1, 2, \ . \ . \ . \ , N.$$

Combining equations (3-16) and (3-13), we obtain

$$H^n = z_1^n\ T\,(x_1^{n-1};\ \theta^n) + G\,(x_1^{n-1};\ \theta^n) + x_2^{n-1},$$

$$n = 1, 2, \ . \ . \ . \ , N.$$

According to equation (3-7), i.e., the stationary condition for optimality, $\bar{\theta}^n$ may be found where

$$\frac{\partial H^n}{\partial \theta^n} = z_1^n\ \frac{\partial T\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n} + \frac{\partial G\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = 0\ .$$

Solving this equation for z_1^n, we obtain

$$z_1^n = -\ \frac{\dfrac{\partial G\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n}}{\dfrac{\partial T\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n}}\ . \qquad (3\text{-}18)$$

Substitution of equation (3-18) into equation (3-17) gives the recurrence relation

$$\frac{\dfrac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n}}{\dfrac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n}} = \frac{\dfrac{\partial G\ (x_1^{n};\ \theta^{n+1})}{\partial \theta^{n+1}}}{\dfrac{\partial T\ (x_1^{n};\ \theta^{n+1})}{\partial \theta^{n+1}}} \cdot \frac{\partial T\ (x_1^n;\ \theta^{n+1})}{\partial x_1^n} - \frac{\partial G\ (x_1^n;\ \theta^{n+1})}{\partial x_1^n},$$

$$n = 1, 2, \ldots, N-1. \tag{3-19}$$

Combining equations (3-15a) and (3-18) gives

$$\frac{\partial G\ (x_1^{N-1};\ \theta^n)}{\partial \theta^N} = 0. \tag{3-20}$$

Making use of the recurrence relation, equation (3-19), along with the performance equations, equation (3-10) and equation (3-20), a number of optimization problems associated with one-dimensional processes can be solved. For processes with a fixed end point x_1^N, condition $z_1^N = 0$ (equation (3-15a)) or equivalently, equation (3-20) is deleted.

4. CASE STUDIES OF SINGLE CONSTRAINT PROBLEMS

EXAMPLE 1: MAXIMIZATION OF THE SYSTEM RELIABILITY SUBJECT TO WEIGHT CONSTRAINTS (LINEAR CONSTRAINT).

Let us consider a system with N sub-systems connected in series. Each sub-system will have some redundant elements connected in parallel. The problem is to determine the optimum number of redundancies at each sub-system so as to maximize the total system reliability subject to the constraint that the total weight in the redundant elements should not exceed a certain fixed amount.

To solve this problem by the discrete maximum principle let us consider each sub-system as a stage and let

θ^n = Number of elements used at nth stage, therefore, number of redundancies employed at the nth stage is $(\theta^n - 1)$,

w^n = Weight of one redundant element in nth stage,

x_1^n = Amount of weight left over after allocating first n stages,

x_2^n = Logarithm of the reliability upto and including first n stages.

This N stage system can be represented by following performance equations.

$$x_1^n = x_1^{n-1} - w^n(\theta^n - 1), \qquad n = 1, 2, \ldots, N, \tag{4-1}$$

$$x_1^0 = W,$$

$$x_1^N \geq 0,$$

$$x_2^n = x_2^{n-1} + \ell n \left(1-(1-R^n)^{\theta^n}\right), \qquad n = 1, 2, \ldots, N, \qquad (4\text{-}2)$$

$$x_2^0 = 0.$$

θ^n is constrained to positive integers only, i.e.,

$$\theta^n \geq 1.$$

The objective function to be maximized is as follows

$$S = \sum_{n=1}^{N} \ell n \left(1-(1-R^n)^{\theta^n}\right) = x_2^N . \qquad (4\text{-}3)$$

x_2^N is logarithm of total system reliability. The system reliability will be maximized if we maximize the logarithm of the reliability.

Comparing equations (4-1) and (4-2) with the equations of one dimensional process, equations (3-10) and (3-11), we see that

$$T(x_1^{n-1};\ \theta^n) = x_1^{n-1} - w^n(\theta^n-1), \qquad (4\text{-}4)$$

$$G(x_1^{n-1};\ \theta^n) = \ell n \left(1-(1-R^n)^{\theta^n}\right). \qquad (4\text{-}5)$$

Differentiating equations (4-4) and (4-5) with respect to x_1^{n-1} and θ^n gives

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = -\ w^n, \qquad (4\text{-}6)$$

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 1, \qquad (4\text{-}7)$$

$$\frac{\partial G(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = \frac{-(1-R^n)^{\theta^n} \ln(1-R^n)}{1-(1-R^n)^{\theta^n}} , \tag{4-8}$$

$$\frac{\partial G(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0 . \tag{4-9}$$

Let us define

$$U^n = 1-R^n .$$

Substituting equations (4-6) through (4-9) into the recurrence relation of one dimensional process, equation (3-19), we obtain

$$\frac{\dfrac{(U^n)^{\theta^n} \ \ln U^n}{1-(U^n)^{\theta^n}}}{-\ w^n} = \frac{\dfrac{(U^{n+1})^{\theta^{n+1}} \ \ln U^{n+1}}{1-(U^{n+1})^{\theta^{n+1}}}}{-\ w^{n+1}} .$$

$$\frac{(U^{n+1})^{\theta^{n+1}} \ \ln U^{n+1}}{1-(U^{n+1})^{\theta^{n+1}}} = \frac{w^{n+1}}{w^n} \ \frac{(U^n)^{\theta^n} \ \ln U^n}{1-(U^n)^{\theta^n}} ,$$

$$\frac{1}{(U^{n+1})^{\theta^{n+1}}} - 1 = \frac{w^n}{w^{n+1}} \ \frac{\ln U^{n+1}}{\ln U^n} \ \frac{1-(U^n)^{\theta^n}}{(U^n)^{\theta^n}} ,$$

or

$$\theta^{n+1} = \frac{\ell n \left(1 + \frac{w^n}{w^{n+1}} \; \frac{\ell n \; U^{n+1}}{\ell n \; U^n} \; \frac{1-(U^n)^{\theta^n}}{(U^n)^{\theta^n}} \right)}{- \ell n \; U^{n+1}} \; . \qquad (4\text{-}10)$$

This is the recurrence relation of the optimal decision. It must be noted that, while θ^n, $n = 1, 2, \ldots, N$ are in reality positive integers, we have assumed that θ^n are continuous variables in obtaining the above recurrence relation. This recurrence relation can be used to obtain a sequence of θ^n which will not violate the weight constraint. The trial and error method, given below is used to obtain sequence of decision variable.

Step 1. Assume $\theta^1 = 1$.

Step 2. Compute $\theta^2, \theta^3, \ldots, \theta^N$ from equation (4-10).

Step 3. Compute $x_1^N = W - \sum_{n=1}^{N} w^n(\theta^n - 1)$.

x_1^N will be in either one of the following situations: (a) greater than zero, (b) equal to zero, (c) less than zero. If it is (a) then go to step 4, if it is (b) then we reached the optimal stage, go to step 6, if it is (c) then go to step 5.

Step 4. Increment θ^1 by 1 and go to step 2.

Step 5. Decrease θ^1 by 0.1 and go to step 2 until x_1^N is again greater than zero, when x_1^N is greater than zero then go to step 6.

Step 6. Solution has reached the optimal stage, convert the solution to an integer solution.

A computer program and the flow diagram for IBM 1620 is given in Appendix A.

NUMERICAL EXAMPLES

A five stage and ten stage systems are considered. However, the computer program and the numerical method developed are for systems with an arbitrary number of stages.

The constants assigned for a five stage problem are given in Table 1. The optimum redundancy obtained is as follows (see Appendix A, Table A3).

$$\theta^1 = 2.6000,$$

$$\theta^2 = 3.8691,$$

$$\theta^3 = 5.2300,$$

$$\theta^4 = 3.5806,$$

$$\theta^5 = 3.0539.$$

Since θ^n, $n = 1, 2, \ldots, 5$ should be positive integers, we obtain

$$\theta^1 = 3,$$

$$\theta^2 = 4,$$

$$\theta^3 = 5,$$

$$\theta^4 = 4,$$

$$\theta^5 = 3.$$

It should be noted that the above figures are the number of units at each stage; to obtain the number of redundancies we should subtract one from each figure, i.e., the number of redundant elements is one less than the number of units in the stage.

The total weight in the redundant elements is equal to 104 units, which is equal to the constraint. This gives the optimal reliability of 0.9850.

Table 1. Constants assigned for 5 stage problem.

n	R^n	w^n	W
1	.90	8	
2	.75	9	
3	.65	6	104
4	.80	7	
5	.85	8	

The constants assigned for a ten stage system are given in Table 2. The optimum redundancy is as given below, (see Appendix A, Table A4).

$\theta^1 = 2.4000,$

$\theta^2 = 1.9227,$

$\theta^3 = 1.6914,$

$\theta^4 = 2.7549,$

$\theta^5 = 3.4449,$

$\theta^6 = 1.7822,$

$\theta^7 = 2.9308,$

$\theta^8 = 1.6604,$

$\theta^9 = 2.3117,$

$\theta^{10} = 1.9496.$

Since, θ^n, $n = 1, 2, \ldots, 10$ should be positive integers, we obtain

$\theta^1 = 2,$

$\theta^2 = 2,$

$\theta^3 = 2,$

$\theta^4 = 3,$

$\theta^5 = 3,$

$\theta^6 = 2,$

$\theta^7 = 3,$

Table 2. Constants assigned for 10 stage problem.

n	R^n	w^n	W
1	.78	17	200
2	.86	19	
3	.88	25	
4	.80	8	
5	.70	8	
6	.90	16	
7	.65	21	
8	.94	11	
9	.79	18	
10	.86	18	

$$\theta^{8} = 2,$$

$$\theta^{9} = 2,$$

$$\theta^{10} = 2.$$

These are the number of units at each stage, and to obtain the redundant elements at each stage we should subtract one from each figure.

The total weight in the redundant elements is 195 units, and thus there is a slack of 2 units in the restraint. This gives the maximum reliability of 0.7853.

EXAMPLE 2: MAXIMIZATION OF THE SYSTEM RELIABILITY SUBJECT TO WEIGHT CONSTRAINTS WHERE MAXIMUM NUMBER OF REDUNDANCIES ARE FIXED. (LINEAR CONSTRAINT).

The system considered is the same as in Example 1 but the maximum number of redundant elements at any stage is limited to some number "m", i.e., the maximum number of elements at any stage can not exceed $m + 1$. The weight constraint remains the same as in the previous problem. The constraint on θ^n can be written as

$$\theta^n \leq m + 1 .$$

The performance equations for this problem remain the same as in the last example; thus, we obtain the same recurrence relation of the optimum decision variable. The only change is the procedure of solving this equation to obtain the desired result in the light of the new constraint on the decision variable. We note that the order of numbering the stages is arbitrary and does not affect the optimal decision. We will make use of this fact in solving the problem.

Step 1. Solve the problem as in example 1, without any consideration of additional constraint.

Step 2. Check whether θ^n, $n = 1, 2, \ldots, N$, is less than $m + 1$. If this is satisfied then the new restriction does not affect our optimal decision and the problem is solved. But if for some $n \leq N$, θ^n is greater than $m + 1$, then go to step 3.

Step 3. If there are p stages for which θ^n is less than "$m+1$" then rearrange the stage number so that all stages for which θ^n is less than "$m+1$" are numbered as $n = 1, 2, \ldots, p$ and the remaining stages for which θ^n is greater than or equal to "$m+1$" are numbered as $n = p+1, p+2, \ldots, N$ and assign $\theta^n = m+1$ for $n = p+1, p+2, \ldots, N$.

Step 4. Consider the new system with p number of stages and the value of W, total weight in redundant elements, changed to

$$W' = W - \sum_{n=p+1}^{N} m w^{n} .$$

Step 5. Go to step 1 with this new system.

The computer program and the flow chart for the IBM 1620 is given in Appendix B.

NUMERICAL EXAMPLES

A three stage and a ten stage problems are solved with a restriction on θ^{n} of maximum equal to five, that is, the maximum number of redundant units at any stage is restricted to four. However, the numerical method developed and the computer program are for systems with an arbitrary number of stages and the number of redundancies can be restricted to any value. It should be noted that this computer program can be used to solve problems of Example 1 by putting in a very large number of redundant elements as a constraint.

The constants assigned for a three stage problem are given in table 3. The optimum redundancy obtained by numerical solution is given below (see Appendix B, Table B3)

$$\theta^{1} = 2.3750,$$

$$\theta^{2} = 5.0000,$$

$$\theta^{3} = 5.0000.$$

Table 3. Constants assigned for 3 stage problem.

n	R^n	w^n	W	θ^n_{max}
1	.95	8		
2	.80	9	75	5
3	.65	7		

As θ^n, $n = 1, 2, 3$ should be positive integers we obtain

$$\theta^1 = 2,$$

$$\theta^2 = 5,$$

$$\theta^3 = 5.$$

It is interesting to note that when there is no restriction on the number of redundancy, we obtain the following optimal policy, (see Table B4).

$$\theta^1 = 2.4000,$$

$$\theta^2 = 4.0065,$$

$$\theta^3 = 5.9780.$$

As θ^n, $n = 1, 2, 3$ should be positive integers we obtain

$$\theta^1 = 2,$$

$$\theta = 4,$$

$$\theta^3 = 6.$$

Since the above figures are the number of elements at each stage, we should subtract one from each of these to obtain the number of redundancy at each stage.

We note that when the number of redundancy is constrained to a maximum of four, we obtain the total weight in redundant elements equal to 72 with system reliability of 0.9919. But when there is no constraint of the number of redundancy total weight in redundant elements is 70 with system reliability of 0.9941. We see that due to the constraint the system reliability is reduced and the total weight in redundant elements increased.

The constants assigned for the ten stage system are given in Table 4. The optimum policy obtained by numerical solution is given below (see Table B5). It should be noted that the stage numbers in Table B5 are not the same as given below. This is because computer has rearranged the stage numbers as explained in step 3 above.

$\theta^1 = 4.1000,$

$\theta^2 = 3.2336,$

$\theta^3 = 2.9049,$

$\theta^4 = 5.0000,$

$\theta^5 = 3.9727,$

$\theta^6 = 2.9041,$

$\theta^7 = 5.0000,$

$\theta^8 = 2.5810,$

$\theta^9 = 3.9606,$

$\theta^{10} = 3.2611.$

As θ^n, $n = 1, 2, \ldots, 10$ should be positive integers, we obtain

$\theta^1 = 4,$

$\theta^2 = 3,$

$\theta^3 = 3,$

$\theta^4 = 5,$

$\theta^5 = 4,$

Table 4. Constants assigned for 10 stage problem

n	R^n	w^n	W	6^nmax.
1	.78	17		
2	.86	19		
3	.88	25		
4	.70	14	475	5
5	.80	15		
6	.90	16		
7	.65	21		
8	.94	11		
9	.79	18		
10	.86	18		

$$\theta^6 = 3,$$

$$\theta^7 = 5,$$

$$\theta^8 = 3,$$

$$\theta^9 = 4,$$

$$\theta^{10} = 3.$$

As indicated before, to obtain the number of redundancy at each stage we subtract one from each of the above figures.

The total weight in redundant units is 468; thus, there is a slack of 7, and this gives the optimum reliability of 0.9782.

Appendix B, Table B6 gives the result of the same problem when θ^n is unconstrained. From the table we note that optimum policy remains the same as before; that is, the additional constraint does not affect the optimum policy for this case.

EXAMPLE 3. MINIMIZATION OF COST OF REDUNDANT ELEMENTS WHILE ACHIEVING A CERTAIN MINIMUM LEVEL OF RELIABILITY (NONLINEAR CONSTRAINT).

Here we consider a system with N sub-systems connected in series with each sub-system having some redundant elements connected in parallel. The system is required to have a certain minimum level of reliability for its efficient functioning. Increasing the number of elements will increase the cost of the system. The problem is to minimize the cost of redundant elements while maintaining the minimum level of reliability.

Consider each subsystem as a stage and let

R = Minimum level of reliability to be achieved,

θ^n = Number of elements in nth sub-system,

θ^n-1= Number of redendant units in nth sub-system,

c^n = Cost per redundant element in nth stage,

x_1^n = Sum of the logarithm of reliability up to and including nth stage,

x_2^n = Cost in redundant elements up to and including nth stage.

This N stage system can be represented by following performance equations.

$$x_1^n = x_1^{n-1} + \ln \{1 - (1-R^n)^{\theta^n}\} , \quad n = 1, 2, . . . , N, \tag{4-12}$$

$$x_1^0 = 0,$$

$$x_1^N \geq \ln(R),$$

$$x_2^n = x_2^{n-1} + c^n(\theta^n-1), \qquad n = 1, 2, . . . , N, \tag{4-13}$$

$$x_2^0 = 0.$$

θ^n should be a positive integer, i.e.,

$$\theta^n \geq 1.$$

We see that the objective function to be minimized is given by

$$S = \sum_{n=1}^{N} x_2^n = \sum_{n=1}^{N} c^n(\theta^n - 1) = x_2^N . \qquad (4\text{-}14)$$

Comparing the performance equations (4-12) and (4-13) with equations of one dimensional processes given by equations (3-10) and (3-11), we find

$$T(x_1^{n-1};\theta^n) = x_1^{n-1} + \ell n \, \{1 - (1-R^n)^{\theta^n}\} , \qquad (4\text{-}15)$$

$$G(x_1^{n-1};\theta^n) = c^n(\theta^n - 1) \qquad (4\text{-}16)$$

Taking the partial derivatives of equations (4-15) and (4-16) with respect to x_1^{n-1} and θ^n yields

$$\frac{\partial T(x_1^{n-1};\theta^n)}{\partial \theta^n} = \frac{-(1-R^n)^{\theta^n} \, \ell n(1-R^n)}{1-(1-R^n)^{\theta^n}} , \qquad (4\text{-}17)$$

$$\frac{\partial T(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 1, \qquad (4\text{-}18)$$

$$\frac{\partial G(x_1^{n-1};\theta^n)}{\partial \theta^n} = c^n, \qquad (4\text{-}19)$$

$$\frac{\partial G(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 0 . \qquad (4\text{-}20)$$

Let us define

$$U^n = 1-R^n \ .$$

Substituting equations (4-17) through (4-20) in the recurrence relation of one dimensional processes given by equation (3-19), we obtain

$$\frac{c^n}{\dfrac{-(U^n)^{\theta^n}\ \ell n U^n}{1-(U^n)^{\theta^n}}} = \frac{c^{n+1}}{\dfrac{-(U^{n+1})^{\theta^{n+1}}\ \ell n U^{n+1}}{1-(U^{n+1})^{\theta^{n+1}}}} ,$$

or

$$\frac{(U^{n+1})^{\theta^{n+1}}\ \ell n U^{n+1}}{1-(U^{n+1})^{\theta^{n+1}}} = \{\frac{c^{n+1}}{c^n}\}\ \{\frac{(U^n)^{\theta^n}\ \ell n U^n}{1-(U^n)^{\theta^n}}\} \ .$$

Comparing this with the recurrence equation for Example 1 we see that both are the same, with w^n replaced by c^n; hence, we obtain the final recurrence relation as

$$\theta^{n+1} = \frac{\ell n\ \{1 + (\dfrac{c^n}{c^{n+1}})(\dfrac{\ell n U^{n+1}}{\ell n U^n})(\dfrac{1}{(U^n)^{\theta^n}} - 1)\}}{-\ell n U^{n+1}} \ . \qquad (4\text{-}21)$$

This is a recurrence relation of the optimal decision variable. It must be noted that, while θ^n, $n = 1, 2, \ldots, N$, are in reality positive integers, we have assumed that θ^n are continuous variables in obtaining the above recurrence relation.

This recurrence relation can be used to obtain the sequence of θ^n which will minimize the cost in redundant elements for any particular level of reliability. The following trial and error method is used to obtain the sequence.

Step 1. Assume $\theta^1 = 1$.

Step 2. Compute $\theta^2, \theta^3, \ldots, \theta^N$ from equation (4-21).

Step 3. Compute $x_1^N = \sum_{n=1}^{N} \ell n \left(1 - (1-R^n)^{\theta^n}\right)$.

x_1^N will be either (a) greater than $\ell n(R)$, (b) equal to $\ell n(R)$, or (c) less than $\ell n(R)$. If it is (a) go to step 5, if it is (b) go to step 7, and if it is (c) go to step 4.

Step 4. Increase θ^1 by one and go to step 2.

Step 5. Decrease θ^1 by 0.1 and go to step 2 until again x_1^N is less than $\ell n(R)$. When x_1^N is less $\ell n(R)$ go to step 6.

Step 6. Increase θ^1 by 0.1 and compute $\theta^2, \theta^3, \ldots, \theta^N$ from equation (4-21).

Step 7. Approximate θ^n, $n = 1, 2, \ldots, N$, to the nearest integer.

A computer program and flow diagram for the IBM 1620 is given in Appendix C.

NUMERICAL EXAMPLES.

A five stage and a ten stage problem is solved. However, the computer program and the numerical method developed is for an arbitrary number of stages.

The constants assigned for a five stage problem are given in Table 5. The optimum redundancies obtained from the above recurrence relation is as follows (see Appendix C, Table C2):

$\theta^1 = 4.4000,$

$\theta^2 = 4.9463,$

$\theta^3 = 3.0511,$

$\theta^4 = 2.6406,$

$\theta^5 = 4.1501.$

As θ^n, $n = 1, \ldots, 5$ should be positive integers, we obtain

$\theta^1 = 4,$

$\theta^2 = 5,$

$\theta^3 = 3,$

$\theta^4 = 3,$

$\theta^5 = 4,$

These figures indicate the number of units at each stage. To obtain the number of redundant units at each stage we subtract one from each figure.

The reliability of the system is 0.9023 where the requirement is 0.9000. This results in a minimum cost of 56.00.

The constants assigned for ten stage problem are given in Table 6. The optimum redundancies obtained is as given below, (see Appendix C, Table C3).

$\theta^1 = 2.8000,$

$\theta^2 = 4.3920,$

$\theta^3 = 1.7346,$

$\theta^4 = 2.2859,$

$\theta^5 = 2.5000,$

Table 5. Constants assigned for 5 stage problem.

n	R^n	c^n	R
1	.65	2.00	
2	.55	3.00	
3	.70	6.00	.9000
4	.75	7.00	
5	.60	4.00	

Table 6. Constants assigned for 10 stage problem.

n	R^n	c^n	$\bar{R}$
1	.75	7.00	
2	.60	4.00	
3	.95	4.00	
4	.80	10.00	
5	.85	4.00	.8500
6	.90	8.00	
7	.80	6.00	
8	.55	3.00	
9	.70	6.00	
10	.65	2.00	

$$\theta^6 = 1.8453,$$

$$\theta^7 = 2.5969,$$

$$\theta^8 = 5.2247,$$

$$\theta^9 = 3.2347,$$

$$\theta^{10} = 4.6136.$$

As θ^n, $n = 1, 2, \ldots, 10$ has to be positive integers, we obtain

$$\theta^1 = 3,$$

$$\theta^2 = 4,$$

$$\theta^3 = 2,$$

$$\theta^4 = 2,$$

$$\theta^5 = 3,$$

$$\theta^6 = 2,$$

$$\theta^7 = 3,$$

$$\theta^8 = 5,$$

$$\theta^9 = 3,$$

$$\theta^{10} = 5.$$

These figures being the number of units at each stage, we get the number of redundant units by subtracting one from each of them.

This policy gives the reliability of the system of 0.8541 against the requirement of 0.8500 and the total minimum cost of 100.00.

EXAMPLE 4: MAXIMIZATION OF SYSTEM RELIABILITY SUBJECT TO A SINGLE NONLINEAR CONSTRAINT.

In this example we will also consider a system with N-stages connected in series, each stage having some redundant unit connected in parallel. The total number of units at each stage is subject to some type of non-linear constraint. Here we consider a separable non-linear constraint. The problem is to maximize the system reliability under this constraint.

Let us consider the constraint of the type

$$g(\theta^n) = \rho^1(\theta^1)^2 + \rho^2(\theta^2)^2 + \ . \ . \ . + \rho^N(\theta^N)^2 \ \leq P \ ,$$

or

$$g(\theta^n) = \sum_{n=1}^{N} \rho^n(\theta^n)^2 \leq P \ ,$$

where ρ^n is constant for nth stage. This constraint can be interpreted as follows

$$\rho^n(\theta^n)^2 = (w^n\theta^n)(c^n\theta^n) \ ,$$

where

w^n = weight per element at nth stage, therefore $(w^n\theta^n)$ will be total weight of nth stage,

c^n = cost per element at nth stage, therefore $(c^n\theta^n)$ will be the cost of nth stage.

This gives us $\rho^n = c^n w^n$.

To apply the discrete maximum principle, the problem is formulated as follows, let

θ^n = Number of elements at nth stage, therefore, number of redundancies employed at nth stage is $(\theta^n - 1)$,

ρ^n = constant of nth stage,

x_1^n = Amount of constraint left over after allocating first n stages,

x_2^n = Logarithm of the reliability upto and including first n stages.

This N-stage system can now be represented by the following performance equations.

$$x_1^n = x_1^{n-1} - \rho^n(\theta^n)^2, \quad n = 1, 2, \ldots, N, \tag{4-22}$$

$$x_1^0 = P,$$

$$x_1^N \geq 0,$$

$$x_2^n = x_2^{n-1} + \ell n \left(1 - (1-R^n)^{\theta^n}\right), \quad n = 1, 2, \ldots, N, \tag{4-23}$$

$$x_2^0 = 0.$$

θ^n is positive integer, i.e.,

$$\theta^n \geq 1.$$

The objective function to be maximized is given by

$$S = \sum_{n=1}^{N} \ell n \left(1 - (1 - R^n)^{\theta^n}\right) = x_2^N . \tag{4-24}$$

This gives the logarithm of maximum reliability. Comparing equations (4-22) and (4-23) with the equations of one dimensional process given by equations (3-10) and (3-11), we obtain

$$T(x_1^{n-1};\theta^n) = x_1^{n-1} - \rho^n(\theta^n)^2, \tag{4-25}$$

$$G(x_1^{n-1};\theta^n) = \ell n \left(1 - (1 - R^n)^{\theta^n}\right) . \tag{4-26}$$

Differentiating these functions with respect to x_1^{n-1} and θ^n gives

$$\frac{\partial T(x_1^{n-1};\theta^n)}{\partial \theta^n} = -2\rho^n\theta^n \ , \tag{4-27}$$

$$\frac{\partial T(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 1 \ , \tag{4-28}$$

$$\frac{\partial G(x_1^{n-1};\theta^n)}{\partial \theta^n} = \frac{-(1 - R^n)^{\theta^n} \ \ln(1 - R^n)}{1 - (1 - R^n)^{\theta^n}} \ , \tag{4-29}$$

$$\frac{\partial G(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 0 \ . \tag{4-30}$$

Let us define

$$U^n = 1 - R^n \ .$$

Substituting equations (4-27) through (4-30) into the recurrence relation of one dimensional process given by equation (3-19), we obtain

$$\frac{\dfrac{(U^n)^{\theta^n} \ \ln(U^n)}{\left(1 - (U^n)^{\theta^n}\right)}}{-2\rho^n\theta^n} = \frac{\dfrac{(U^{n+1})^{\theta^{n+1}} \ \ln(U^{n+1})}{\left(1 - (U^{n+1})^{\theta^{n+1}}\right)}}{-2\rho^{n+1}\theta^{n+1}} \ ,$$

or

$$\frac{\left(1 - (U^n)^{\theta^n} \ \rho^n\theta^n\right)}{(U^n)^{\theta^n} \ \ln(U^n)} = \frac{\left(1 - (U^{n+1})^{\theta^{n+1}}\right)\rho^{n+1} \ \theta^{n+1}}{(U^{n+1})^{\theta^{n+1}} \ \ln(U^{n+1})} \ .$$

This reduces to

$$\frac{\theta^{n+1}\left(1-(U^{n+1})^{\theta^{n+1}}\right)}{(U^{n+1})^{\theta^{n+1}}} = \left(\frac{\rho^{n}}{\rho^{n+1}}\right)\left(\frac{\ell n\, U^{n+1}}{\ell n U^{n}}\right)\left(\frac{1-(U^{n})^{\theta^{n}}}{(U^{n})^{\theta^{n}}}\right)\theta^{n} .$$

Noting that all terms at the right hand side are known if θ^n is known, and we need to find θ^{n+1}, let

$$A^{n} = \left(\frac{\rho^{n}}{\rho^{n+1}}\right)\left(\frac{\ell n U^{n+1}}{\ell n U^{n}}\right)\left(\frac{1-(U^{n})^{\theta^{n}}}{(U^{n})^{\theta^{n}}}\right)\theta^{n} ,$$

we obtain

$$\frac{\theta^{n+1}\left(1-(U^{n+1})^{\theta^{n+1}}\right)}{(U^{n+1})^{\theta^{n+1}}} = A^{n} ,$$

$$\theta^{n+1}\left(1-(U^{n+1})^{\theta^{n+1}}\right) = A^{n}(U^{n+1})^{\theta^{n+1}} ,$$

or

$$\theta^{n+1} = \left(A^{n}+\theta^{n+1}\right)\ (U^{n+1})^{\theta^{n+1}} . \qquad (4\text{-}31)$$

The equation is solved by Newton's method. It is worth noting that, while θ^n, $n = 1, 2, \ldots, N$ are in reality positive integers it is assumed that θ^n are continuous variables in obtaining the above recurrence relation. The above equation may be written as

$$f(\theta^{n+1}) = \theta^{n+1} - (A^{n}+\theta^{n+1})\ (U^{n+1})^{\theta^{n+1}} = 0 . \qquad (4\text{-}32)$$

As U^{n+1} is less than one, it can be easily seen that $f(\theta^{n+1})$ will be a monotonically increasing function of θ^{n+1}. At $\theta^{n+1} = 0$ we find that $f(\theta^{n+1})$ is negative. As we increase the value of θ^{n+1}, this function will assume a higher and higher value, that is, starting from a negative value, $f(\theta^{n+1})$ will pass through zero and then become positive. The nature of this curve is shown in Fig. 5. As the curve is well behaved, Newton's method can be applied to obtain the solution. The stepwise procedure is given below.

Step 1. Assume $\theta^1 = 1$, and $n = 1$.

Step 2. Assume $(\theta^{n+1})_1 = 1$.

Step 3. Compute A^n.

Step 4. Compute

$$f\{(\theta^{n+1})_1\} = (\theta^{n+1})_1 - \left(A^n + (\theta^{n+1})_1\right)(U^{n+1})^{(\theta^{n+1})_1} ,$$

and

$$f'\{(\theta^{n+1})_1\} = 1 - \left[(U^{n+1})^{(\theta^{n+1})_1} + \{(\theta^{n+1})_1 + A^n\}(U^{n+1})^{(\theta^{n+1})_1} \ln U^{n+1}\right]. \tag{4-32a}$$

Step 5. Compute a new trial value for θ^{n+1} from following equation

$$(\theta^{n+1})_2 = (\theta^{n+1})_1 - \frac{f\left((\theta^{n+1})_1\right)}{f'\left((\theta^{n+1})_1\right)} .$$

Step 6. Check if

$$|(\theta^{n+1})_2 - (\theta^{n+1})_1| \leq (E_R)_{max} .$$

$(E_R)_{max}$ will depend on the accuracy of the result desired. If this is satisfied go to step 7, if this is not satisfied go to step 4 with new value of θ^{n+1}.

Step 7. The value of θ^{n+1} is obtained an optimum one corresponding to the assumed value of θ^1. Increase n by one and go to step 2, until n becomes greater than N; when n becomes greater than N go to step 8.

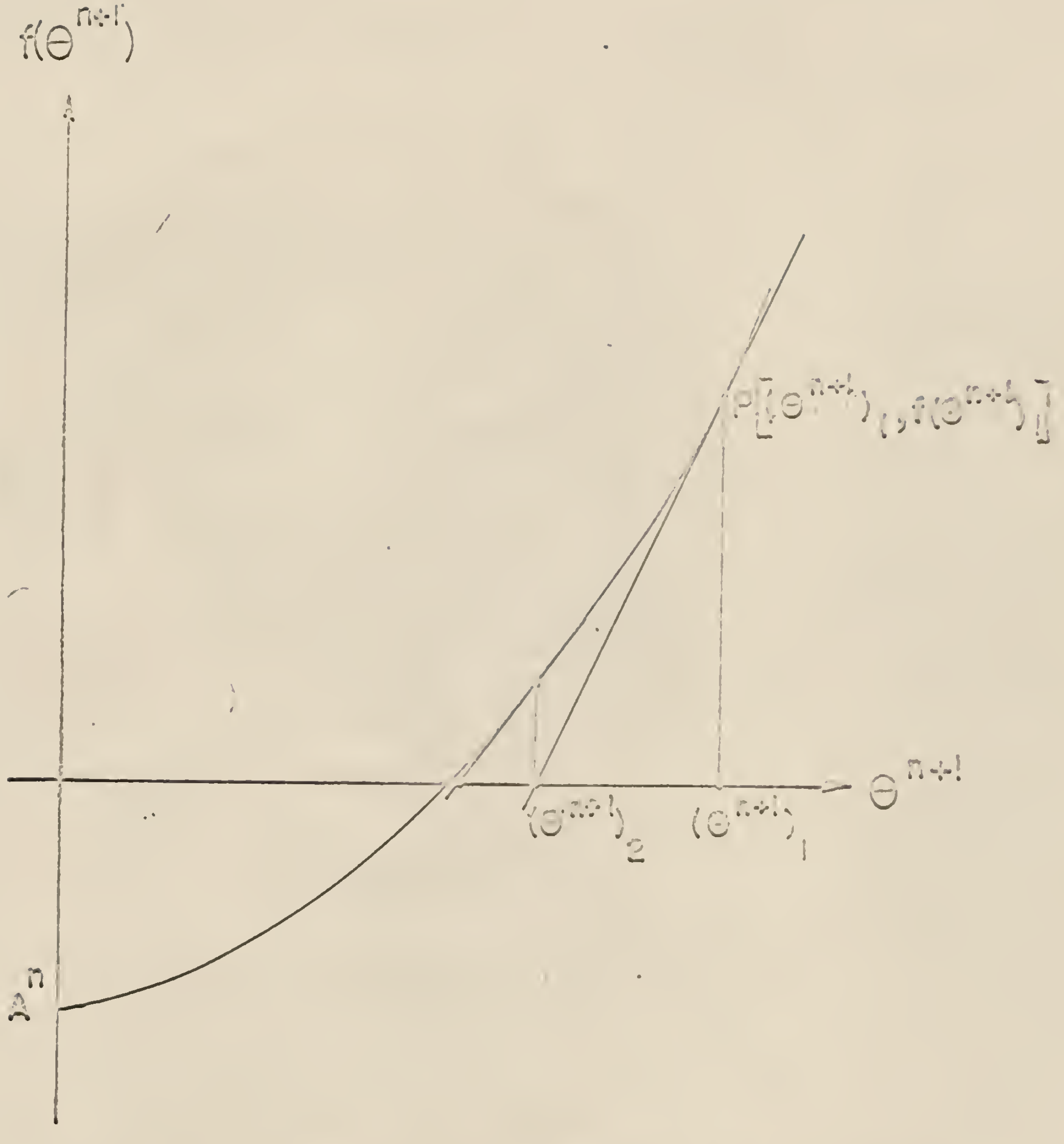

Fig. 5. Application of Newton's method for solution of equation (4-32).

Step 8. Compute

$$x_1^N = P - \sum_{n=1}^{N} \rho^n (\theta^n)^2,$$

Now one of the following condition will exist. (a) x_1^N will be greater than zero, (b) equal to zero, (c) less than zero. If it is (a) go to step 9, if it is (b) we have reached the optimal stage, go to step 11, if it is (c) go to step 10.

Step 9. Increment θ^1 by one, make $n = 1$ again and go back to step 3.

Step 10. Decrease θ^1 by 0.1 make $n = 1$ again and go to step 3 until x_1^N is again greater than zero, when x_1^N is greater than zero then go to step 11.

Step 11. The solution has reached the optimal stage; round off the solution to an integer.

A computer program and flow diagram for the IBM 1620 is given in Appendix D.

NUMERICAL EXAMPLES

Three stage and eight stage problems are solved. However, the numerical method developed and the computer program are for systems with an arbitrary number of stages.

The constants assigned for a three stage problem are given in Table 7. The optimum redundancy obtained is as follows (see Appendix D, Table D3):

$$\theta^1 = 1.9000,$$

$$\theta^2 = 1.3134,$$

$$\theta^3 = 2.7603.$$

Since θ^n, $n = 1, 2, 3$ should be positive integers, we obtain

Table 7. Constants assigned for 3 stage problem.

n	R^n	p^n	P
1	.85	5.00	50
2	.95	8.00	
3	.75	2.00	

$\theta^1 = 2,$

$\theta^2 = 1,$

$\theta^3 = 3.$

Thus the number of redundant units can be obtained by subtracting one from each of the above figures.

From the result we find that 46 units out of 50 units of resource are used, and the total system reliability achieved is 0.9141.

The constants assigned for the eight stage problem are given in Table 6. The optimum redundancy obtained is as follows (see Appendix D, Table D4):

$\theta^1 = 2.3000,$

$\theta^2 = 1.5748,$

$\theta^3 = 3.3109,$

$\theta^4 = 5.3310,$

$\theta^5 = 3.2511,$

$\theta^6 = 3.6317,$

$\theta^7 = 3.1578,$

$\theta^8 = 2.5341.$

Since θ^n, $n = 1, 2, \ldots, 8$ should be positive integers, we obtain

$\theta^1 = 2,$

$\theta^2 = 2,$

$\theta^3 = 3,$

$\theta^4 = 5,$

Table 8. Constants assigned for 8 stage problem.

n	R^n	p^n	P
1	.85	5.00	
2	.95	8.00	
3	.75	2.00	
4	.55	1.00	
5	.60	7.00	300
6	.65	3.00	
7	.70	4.00	
8	.80	5.00	

$$\theta^5 = 3,$$

$$\theta^6 = 4,$$

$$\theta^7 = 3,$$

$$\theta^8 = 3.$$

The above figures being the number of elements at each stage, we can obtain the number of redundant units at each stage by subtracting one from each of them.

Here we note that 287 units out of 300 units of resource is used which gives the optimum reliability of 0.8384.

5. OPTIMIZATION OF PROBLEMS INVOLVING SEPARABLE FUNCTIONS

In this section optimization of problems with a separable objective function and separable and monotonically increasing constraints is considered. A separable and monotonically increasing function can be expressed as

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i)$$

and

$$f(x_1, x_2, \ldots, x_i+1, \ldots, x_n) > f(x_1, x_2, \ldots, x_n)$$

where $f_i(x_i)$ is only a function of x_i. Many transformations are available to reduce the function to a separable form. In previous examples, the function of total system reliability is reduced to a separable form by taking logarithm.

A general problem with separable functions may be stated as follows:

Optimize

$$S = \sum_{n=1}^{N} f^n(\theta^n), \tag{5-1}$$

subject to

$$\sum_{n=1}^{N} g_i^n(\theta^n) \leq b_i, \qquad i = 1, 2, \ldots, s . \tag{5-2}$$

There are various methods available to solve such types of problems, such as integer programming, the gradient method, quadratic programming, tangent approximation etc. In this section the discrete maximum principle is applied to such problems.

To solve the above problem by the method of the discrete maximum principle, we introduce (s+1) state variables, s state variables for s constraints, one for the objective function. Let

x_i^n = ith resource consumed corresponding to ith constraint up to and including first n stages, i = 1, 2, ..., s,

x_{s+1}^n = the accumulated objective function up to and including the first n stages.

The performance equations for this N-stage process may be written as

$$x_i^n = x_i^{n-1} + g_i^n(\theta^n), \qquad n = 1, 2, \ldots, N, \qquad i = 1, 2, \ldots, s, \tag{5-3}$$

$$x_i^0 = 0,$$

$$x_i^N \leq b_i,$$

$$x_{s+1}^n = x_{s+1}^{n-1} + f^n(\theta^n), \quad n = 1, 2, \ldots, N, \tag{5-4}$$

$$x_{s+1}^0 = 0.$$

The objective function to be optimized is

$$S = \sum_{n=1}^{N} f^n(\theta^n)$$

$$= x_{s+1}^N = \sum_{i=1}^{s+1} c_i x_i^N . \tag{5-5}$$

Equation (5-5) gives

$$c_i = 0, \quad i = 1, 2, \ldots, s, \tag{5-6a}$$

$$c_{s+1} = 1. \tag{5-6a}$$

The Hamiltonian and the adjoint variables can be written as

$$H^n = \sum_{i=1}^{s+1} z_i^n x_i^n$$

$$= \sum_{i=1}^{s} z_i^n \left(x_i^{n-1} + g_i^n(\theta^n)\right) + z_{s+1}^n \left(x_{s+1}^{n-1} + f^n(\theta^n)\right), \qquad (5\text{-}7)$$

$$n = 1, 2, \ldots, N,$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} = z_i^n, \qquad n = 1, 2, \ldots, N; \quad i = 1, 2, \ldots, s, \qquad (5\text{-}8)$$

$$z_{s+1}^{n-1} = \frac{\partial H^n}{\partial x_{s+1}^{n-1}} = z_{s+1}^n, \qquad n = 1, 2, \ldots, N. \qquad (5\text{-}9a)$$

From equation (5-6b)

$$z_{s+1}^N = 1. \qquad (5\text{-}9b)$$

Equations, (5-9a) and (5-9b) yield

$$z_{s+1}^n = 1, \qquad n = 1, 2, \ldots, N.$$

This reduces equation (5-7) to

$$H^n = \sum_{i=1}^{s} z_i^n \left(x_i^{n-1} + g_i^n(\theta^n)\right) + x_{s+1}^{n-1} + f^n(\theta^n). \qquad (5\text{-}10)$$

Differentiating equation (5-10) with respect to θ^n and equating to zero, we obtain

$$\frac{\partial H^n}{\partial \theta^n} = 0 = \sum_{i=1}^{s} z_i^n \frac{\partial g_i^n(\theta^n)}{\partial \theta^n} + \frac{\partial f^n(\theta^n)}{\partial \theta^n},$$

or

$$\sum_{i=1}^{s} z_i^n \frac{\partial g_i^n(\theta^n)}{\partial \theta^n} + \frac{\partial f^n(\theta^n)}{\partial \theta^n} = 0 . \qquad (5\text{-}11)$$

Now assume that the jth constraint is active and the rest are free. This means the end condition corresponding to the jth constraint is fixed and the rest of them are free. From equations (3-9) and (5-6a), we have

$$z_i^N = c_i = 0, \qquad i = 1, 2, \ldots, s, \quad i \neq j. \qquad (5\text{-}12)$$

From equation (5-8) and (5-12) we obtain

$$z_i^n = 0, \qquad i = 1, 2, \ldots, s, \quad i \neq j, \quad n = 1, 2, \ldots, N.$$

Therefore, equation (5-11) reduces to

$$z_j^n \frac{\partial g_i^n(\theta^n)}{\partial \theta^n} + \frac{\partial f^n(\theta^n)}{\partial \theta^n} = 0. \qquad (5\text{-}13)$$

The procedure for solving the problem may be described in following steps.

Step 1. Assuming some value for θ^1 in equation (5-13), we obtain z_j^1. However, equation (5-8) gives

$$z_j^1 = z_j^n, \qquad n = 1, 2, \ldots, N.$$

Step 2. Values of z_j^n obtained are used to find θ^n, $n = 2, 3, \ldots, N$ from equation (5-13).

Step 3. Check to see if any of the given constraints are violated. If no constraint is violated then assume a higher value for θ^1 and go to step 1. If the active constraint is violated then assume a smaller value for θ^1 and go to step 1. If the active constraint is not violated and some other constraint k is violated, then go to step 4. If the active constraint reaches its limit and none of the other constraints are violated then go to step 5.

Step 4. Replace constraint j by constraint k, i.e., now constraint k is active and the rest of them are free. Therefore in equation (5-13) and steps 1, 2 and 3 replace j by k and go back to step 1.

Step 5. We have reached the optimal solution with the constraint in action as the only active constraint on the system.

In the above iterative process we may come across a case where considering jth constraint in action violates the kth constraint before the jth constraint reaches its limit and considering the kth constraint in action violates the jth constraint before the kth constraint reaches its limit. This indicates that the optimum lies at the intersection of the constraints j and k. Under such condition we obtain the following equations.

When the constraints j and k are both active we find that

$$z_i^n = 0, \qquad i = 1, 2, \ldots, s \qquad i \neq j, k. \tag{5-14}$$

This reduces equation (5-11) to

$$z_j^n \frac{\partial g_j^n(\theta^n)}{\partial \theta^n} + z_k^n \frac{\partial g_k^n(\theta^n)}{\partial \theta^n} + \frac{\partial f^n(\theta^n)}{\partial \theta^n} = 0 \tag{5-15}$$

The stepwise procedure to solve this problem is given below.

Step 1. Assuming some value for θ^1 in equation (5-15), we obtain a relation between z_j^1 and z_k^1 which may be written as

$$z_j^1 = h(z_k^1) \tag{5-16}$$

but from equation (5-8) we know that

$$z_j^1 = z_j^n, \qquad n = 2, 3, \ldots, N,$$

and

$$z_k^1 = z_k^n, \qquad n = 2, 3, \ldots, N.$$

This reduces equation (5-16) to

$$z_j^n = h(z_k^n), \qquad n = 1, 2, \ldots, N.$$

Substituting this into equation (5-15) we get

$$h(z_k^n) \left(\frac{\partial g_j^n(\theta^n)}{\partial \theta^n}\right) + z_k^n \left(\frac{\partial g_k^n(\theta^n)}{\partial \theta^n}\right)$$

$$+ \frac{\partial f^n(\theta^n)}{\partial \theta^n} = 0. \qquad (5\text{-}17)$$

Step 2. Assuming some value for θ^2 in equation (5-17), we obtain z_k^2 and z_k^n since $z_k^2 = z_k^n$, $n = 1, 2, \ldots, N$.

Step 3. The value of z_k^n obtained is used in equation (5-17) to calculate θ^n, $n = 3, 4, \ldots, N$.

Step 4. Check if constraint j or k is violated. There will be four conditions that may exists at this step.

(1) Neither of the two constraints is violated,

(2) the constraint k reaches its limit before constraint j reaches its limit,

(3) the constraint j is violated before the constraint k reaches its limit,

(4) both the constraints reach their limits.

If condition (1) exists then go to step 5,

if condition (2) exists then go to step 6,

if condition (3) exists then go to step 7,

if condition (4) exists then go to step 8.

Step 5. Increment θ^2 to some higher value and go to step 2.

Step 6. Increase θ^1 to some higher value and go to step 1.

Step 7. Decrease θ^1 by some value and go to step 1.

Step 8. The optimum has been reached; terminate the process.

If there are more than two constraints active at one time, the above procedure can be extended to deal with that situation.

6. CASE STUDIES OF MULTIPLE CONSTRAINTS PROBLEMS.

EXAMPLE 5. MAXIMIZATION OF SYSTEM RELIABILITY SUBJECT TO MULTIPLE LINEAR CONSTRAINTS

In this example multiple linear constraints are considered. The linear constraints are always separable. It is seen in section 4 that the objective function can be reduced to separable form by taking logarithm of the total system reliability. Therefore in this problem, the constraints and the objective function both being of separable form, can be described by equations (5-2) and (5-1) respectively. Thus this problem is a special case of a general class of optimization problems discussed in section 5. First we will consider a problem with two stages and three constraints. The method will be illustrated by hand computation in detail. The method developed is for an N-stage system and can be applied for any number of constraints. A general computer program is given in Appendix E. The numerical solutions to five and ten stage problems will be presented.

In the maximization of system reliability, increasing the number of redundant units increases the total weight, the cost involved and the volume occupied. Usually these three resources are limited and some type of constraints on the number of units that can be employed at each stage are presented in such problems.

Let

c^n = Cost of one unit at nth stage,

w^n = Weight of one unit at nth stage,

v^n = Volume of one unit at nth stage,

θ^n = Number of units at nth stage.

Then constraints can be formulated as

(1) The cost constraint

$$g_1 = \sum_{n=1}^{N} c^n \theta^n \leq C .$$

(2) The weight constraint

$$g_2 = \sum_{n=1}^{N} w^n \theta^n \leq W.$$

(3) The volume constraint

$$g_3 = \sum_{n=1}^{N} v^n \theta^n \leq V.$$

The problem is to maximize the system reliability subject to the above constraints.

To put the problem into the discrete maximum principle format, let us define the state variables. Let

x_1^n = Cost of elements up to and including first n stages,

x_2^n = Amount of weight allocated to first n stages,

x_3^n = Volume occupied at the end of first n stage,

x_4^n = Logarithm of reliability up to and including first n stages.

The problem will be formulated as an N stage problem. The performance equations for this N stage problem are represented as follows

$$x_1^n = x_1^{n-1} + c^n \theta^n, \quad n = 1, 2, \ldots, N, \tag{6-1}$$

$$x_1^0 = 0,$$

$$x_1^N \leq C,$$

$$x_2^n = x_2^{n-1} + w^n \theta^n, \qquad n = 1, 2, \ldots, N, \tag{6-2}$$

$$x_2^0 = 0,$$

$$x_2^N \leq W,$$

$$x_3^n = x_3^{n-1} + v^n \theta^n, \qquad n = 1, 2, \ldots, N, \tag{6-3}$$

$$x_3^0 = 0,$$

$$x_3^N \leq V,$$

$$x_4^n = x_4^{n-1} + \ell n \left(1 - (1 - R^n)^{\theta^n}\right), \qquad n = 1, 2, \ldots, N, \tag{6-4}$$

$$x_4^0 = 0.$$

θ^n is constrained to be positive integers only, that is,

$$\theta^n \geq 1.$$

The objective function to be maximized is

$$S = \sum_{n=1}^{N} \ell n \left(1 - (1 - R^n)^{\theta^n}\right)$$

$$= x_4^N = \sum_{i=1}^{4} c_i x_i^N . \tag{6-5}$$

Therefore we obtain

$$c_i = 0, \qquad i = 1, 2, 3, \tag{6-6}$$

$$c_4 = 1.$$

The Hamiltonian function and the adjoint variables can be written as

$$H^n = \sum_{i=1}^{4} z_i^n x_i^n$$

$$= z_1^n \left(x_1^{n-1} + c^n \theta^n\right) + z_2^n \left(x_2^{n-1} + w^n \theta^n\right)$$

$$+ z_3^n \left(x_3^{n-1} + v^n \theta^n\right) + z_4^n \left(x_4^{n-1} + \ln \{1 - (1 - R^n)^{\theta^n}\}\right) \quad , \qquad (6\text{-}7)$$

$n = 1, 2, \ldots, N,$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n, \qquad (6\text{-}8)$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n, \qquad (6\text{-}9)$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_3^n, \qquad (6\text{-}10)$$

$$z_4^{n-1} = \frac{\partial H^n}{\partial x_4^{n-1}} = z_4^n, \qquad (6\text{-}11)$$

$$z_4^N = c_4 = 1. \qquad (6\text{-}11a)$$

From equations (6-11) and (6-11a) we obtain

$$z_4^n = 1, \qquad n = 1, 2, \ldots, N.$$

Hence equation (6-7) reduces to

$$H^n = z_1^n \left(x_1^{n-1} + c^n \theta^n\right) + z_2^n \left(x_2^{n-1} + w^n \theta^n\right)$$

$$+ z_3^n \left(x_3^{n-1} + v^n \theta^n\right) + x_4^{n-1} + \ln \left(1 - (1 - R^n)^{\theta^n}\right) . \qquad (6\text{-}12)$$

Differentiating equation (6-12) with respect to θ^n and equating to zero, we obtain

$$\frac{\partial H^n}{\partial \theta^n} = 0 = z_1^n c^n + z_2^n w^n + z_3^n v^n + \left(\frac{-(1 - R^n)^{\theta^n} \ln(1 - R^n)}{1 - (1 - R^n)^{\theta^n}}\right) \qquad (6\text{-}13)$$

Let us consider now that the first constraint is the active one and the rest are free. Then from equation (6-6)

$$z_2^N = c_2 = 0,$$

which gives

$$z_2^n = 0, \quad n = 1, 2, \ldots, N,$$

and

$$z_3^N = c_3 = 0,$$

which gives

$$z_2^n = 0, \quad n = 1, 2, \ldots, N,$$

Therefore equation (6-13) reduces to

$$\frac{\partial H^n}{\partial \theta^n} = 0 = z_1^n c^n + \left(\frac{-(1 - R^n)^{\theta^n} \ln(1 - R^n)}{1 - (1 - R^n)^{\theta^n}}\right) . \qquad (6\text{-}14)$$

Let $U^n = 1 - R^n$, and substituting U^n in equation (6-14), we obtain

$$\frac{\partial H^n}{\partial \theta^n} = 0 = z_1^n c^n + \left(\frac{-(U^n)^{\theta^n} \ \ln U^n}{1 - (U^n)^{\theta^n}}\right) \quad . \tag{6-14a}$$

Solving equation (6-14a) for z_1^n yields

$$z_1^n = \frac{1}{c^n} \left(\frac{(U^n)^{\theta^n} \ \ln U^n}{1 - (U^n)^{\theta^n}}\right) \quad . \tag{6-15}$$

Solving equation (6-14a) for θ^n, we obtain

$$\frac{z_1^n c^n}{\ln U^n} = \frac{(U^n)^{\theta^n}}{1-(U^n)^{\theta^n}} \quad ,$$

or

$$\frac{1}{(U^n)^{\theta^n}} = \frac{z_1^n c^n + \ln U^n}{z_1^n c^n} \quad ,$$

or

$$\theta^n = \frac{\ln\left(\frac{z_1^n c^n}{z_1^n c^n + \ln U^n}\right)}{\ln U^n} \tag{6-16}$$

The procedure is demonstrated by hand computations in detail for a two stage problem. The constants are given in Table 9.

Let $\theta^1 = 1$, then from equation (6-15), we obtain

$$z_1^1 = \frac{1}{c^1} \left(\frac{(U^1)^1 \ln U^1}{1-(U^1)^1} \right)$$

$$= \frac{1}{9} \left(\frac{(1 - 0.9) \ln(1 - 0.9)}{1 - (1 - 0.9)} \right)$$

$$= - \frac{2.3}{81} .$$

From equation (6-8), we have

$$z_1^1 = z_1^2 .$$

Therefore

$$z_1^2 = - \frac{2.3}{81} .$$

This gives, from equation (6-16)

$$\theta^2 = \frac{\ln \left(\dfrac{z_1^2 c^2}{z_1^2 c^2 + \ln U^2} \right)}{\ln U^2}$$

$$= \frac{\ln \left(\dfrac{\frac{2.3}{81} \times 5}{\frac{2.3 \times 5}{81} + \ln(.25)} \right)}{\ln(.25)}$$

$$= 1.57.$$

Checking if any constraint is violated

Table 9. Constants assigned for 2 stage problem

n	R^n	c^n	C	w^n	W	v^n	V
1	0.9	9		5		1	
2	0.75	5	50	9	45	1	6

$$x_1^N = 9 + 5 \times 1.57$$

$$= 16.85 < 50,$$

$$x_2^N = 5 + 9 \times 1.57$$

$$= 19.03 < 45,$$

$$x_3^N = 1 + 1.57$$

$$= 2.57 < 6.$$

Thus no constraint is violated; therefore, let

$$\theta^1 = 2 .$$

This gives, from equation (6-15),

$$z_1^1 = \frac{-2\cdot 3}{891} = z_1^2 .$$

We obtain, from equation (6-16),

$$\theta^2 = 3.38 .$$

Checking if any constraint is violated

$$x_1^N = 9 \times 2 + 5 \times 3.38$$

$$= 34.90 < 50,$$

$$x_2^N = 5 \times 2 + 9 \times 3.38$$

$$= 40.42 < 45,$$

$$x_3^N = 2 + 3.38$$

$$= 5.38 < 6.$$

As no constraint is violated, we take a new trial value for $\theta^1 = 3$.

This gives

$$z_1^1 = \frac{-\ 2\cdot 3}{9 \times 999} .$$

Therefore

$$\theta^2 = 5.04 .$$

Checking if any constraint is violated

$$x_1^N = 9 \times 3 + 5 \times 5.04$$

$$= 42.2 < 50,$$

$$x_2^N = 5 \times 3 + 9 \times 5.04$$

$$= 60.2 > 45,$$

$$x_3^N = 3 + 5.04$$

$$= 8.04 > 6 .$$

Thus the last two constraints are violated but not the one considered to be active. So we make this constraint free and assume that the third constraint is active. This gives, from equation (6-6)

$$z_1^N = c_1 = 0,$$

$$z_1^n = 0, \quad n = 1, 2, . . . , N,$$

$$z_2^N = c_2 = 0,$$

$$z_2^n = 0, \quad n = 1, 2, . . . , N,$$

and

$$z_3^n \neq 0 .$$

Substituting the values of z_1^n and z_2^n into equation (6-13) and solving for z_3^n and θ^n, we obtain

$$z_3^n = \frac{1}{v^n} \left(\frac{(U^n)^{\theta^n} \ \ell n U^n}{1 - (U^n)^{\theta^n}}\right) , \qquad (6\text{-}17)$$

and

$$\theta^n = \frac{\ln\left(\dfrac{z^n v^n}{z_3^n v^n + \ln U^n}\right)}{\ln U^n} \quad . \qquad (6\text{-}18)$$

Now let $\theta^1 = 2$ which gives, from equation (6-17)

$$z_3^1 = -\frac{2\cdot 3}{99} \; .$$

This gives, from equation (6-18),

$$\theta^2 = 2.89 \; .$$

Checking if any constraint is violated

$$x_1^N = 9 \times 2 + 5 \times 2.89$$

$$= 32.45 < 50,$$

$$x_2^N = 5 \times 2 + 9 \times 2.89$$

$$= 36.1 < 45,$$

$$x_1^N = 2 + 2.89$$

$$= 4.89 < 6 \; .$$

None of the constraints are violated, so let the new trial value of θ^1 be 3 which gives

$$z_3^1 = -\frac{2\cdot 3}{999} \; .$$

We obtain

$$\theta^2 = 4.62 \; .$$

Substituting in the constraints we will find that both constraints one and two are violated but not the constraint three. Out of the two constraints violated the constraint assumed active is also violated. Therefore to know which constraint is violated first, we take a new trial value for θ^1 less than the previous value. Let $\theta^1 = 2.5$ which gives

$$z_3^1 = -\frac{2.3}{311.5} ,$$

and

$$\theta^2 = 3.76 .$$

Here it is found that both constraints 1 and 2 are violated. Let the new trial value of θ^1 be 2.4 which gives

$$z_3^1 = -\frac{2.3}{249} ,$$

and

$$\theta^2 = 3.6 .$$

None of the constraints are violated by this assumption and the third constraint is active. Therefore, the optimum policy is

$$\theta^1 = 2.4,$$

$$\theta^2 = 3.6 .$$

It should be noted that in all calculations θ^n are assumed to be continuous variables, while in fact they are positive integers only. This gives

$$\theta^1 = 2,$$

$$\theta^2 = 4.$$

This policy is not violating any of the three constraints and hence is the solution.

NUMERICAL EXAMPLES:

Five stage and ten stage problems with three constraints are solved. However, the numerical method developed and the computer program are for systems with an arbitrary number of stages and there can be any number of constraints.

The constants assigned for five stage problem are given in Table 10. The optimum redundancy obtain is as follows, (see Appendix E, Table E3).

$$\theta^1 = 2.9000,$$

$$\theta^2 = 4.6120,$$

$$\theta^3 = 5.0561,$$

$$\theta^4 = 3.7182,$$

$$\theta^5 = 3.2408.$$

Since θ^n, $n = 1, 2, \ldots, 5$ should be positive integers, we obtain

$$\theta^1 = 3,$$

$$\theta^2 = 5,$$

$$\theta^3 = 5,$$

$$\theta^4 = 4,$$

$$\theta^5 = 3.$$

The number of redundant elements at each stage can be obtained by subtracting one from each of the above figure.

From the result we find that the total weight of the system is 123 units with 9 units of slack, the cost of the system is 151 units with 4 units of slack and volume of the system is 49 units with 1 unit of slack. This results in a system reliability of 0.9878.

Table 10. Constants assigned for 5 stage problem

n	R^n	c^n	C	w^n	W	v^n	V
1	.90	8		3		3	
2	.75	9		4		2	
3	.65	6	155	9	132	4	50
4	.80	7		7		1	
5	.85	8		7		2	

The constants assigned for the ten stage problem are given in Table 11. The optimum policy obtained is as follows (see Appendix B, Table [illegible]).

$\theta^1 = 2.8000$,

$\theta^2 = 3.6784$,

$\theta^3 = 1.4220$,

$\theta^4 = 2.6214$,

$\theta^5 = 2.2735$,

$\theta^6 = 1.9286$,

$\theta^7 = 2.3740$,

$\theta^8 = 4.7186$,

$\theta^9 = 3.0632$,

$\theta^{10} = 3.3859$.

Since, θ^n, $n = 1, 2, \ldots, 10$ should be positive integers, we obtain

$\theta^1 = 3$,

$\theta^2 = 4$,

$\theta^3 = 1$,

$\theta^4 = 3$,

$\theta^5 = 2$,

$\theta^6 = 2$,

$\theta^7 = 2$,

$\theta^8 = 5$,

Table 11. Constants assigned for 10 stage problem

n	R^n	c^n	C	w^n	W	v^n	V
1	.75	7		17		1	
2	.60	4		19		1	
3	.95	4		25		1	
4	.80	10		14		1	
5	.85	4	175	15	475	1	
6	.90	8		16		1	40
7	.80	6		21		1	
8	.55	3		11		1	
9	.70	6		18		1	
10	.65	2		18		1	

$$\theta^9 = 3,$$

$$\theta^{10} = 3.$$

The number of redundant elements at any stage may be obtained by subtracting one from each of the above figures.

The result shows that the total weight of the system is 461 units with a slack of 14 units, the cost is 146 units with a slack of 29 units and the volume is 28 units with a slack of 12 units. This results in a total system reliability of 0.7676.

EXAMPLE 6. MAXIMIZATION OF SYSTEM RELIABILITY SUBJECT TO MULTIPLE NON-LINEAR CONSTRAINTS

In this example the method described in section 5 will be applied to the problem of maximizing system reliability subject to non-linear constraints. The method developed is for an N-stage system. N stages are connected in series. Each stage can have different number of redundant units.

In the last example we assume that increase in cost and weight due to redundant elements is directly proportional to the number of units at each stage, that is, weight and cost increases linearly with the number of elements at each stage. This is not necessarily a correct assumption. As the number of units is increased at each stage it requires more connecting equipment. As the number of elements at each stage is increased, the cost and weight may increase exponentially due to the additional hook up required to connect these elements. Therefore three types of constraints will be considered.

Let

c^n = cost per element at nth stage,

w^n = weight per element at nth stage,

v^n = volume per element at nth stage,

θ^n = number of elements at nth stage.

(1) The constraint of the type

$$g_1 = \sum_{n=1}^{N} p^n (\theta^n)^2 \leq P,$$

where $p^n = w^n v^n$, the product of weight per unit and volume per unit at nth stage. This is similar to one given in example 4.

(2) The cost constraint

$$g_2 = \sum_{n=1}^{N} c^n\left(\theta^n + \exp(\theta^n/4)\right) \le C,$$

where $c^n\theta^n$ will be the cost of units at nth stage and $c^n e^{\theta^n/4}$ will be the additional cost of the hook up (13).

(3) The weight constraint

$$g_3 = \sum_{n=1}^{N} w^n\theta^n \exp(\theta^n/4) \le W,$$

where $w^n\theta^n$ will be the weight of units at nth stage. This is increased by a factor $\exp(\theta^n/4)$ due to the weight of the hook up.

The problem is to maximize total system reliability subject to the above constraints. To formulate the problem as a discrete maximum principle problem, we define state variables as follows.

$$x_1^n = x_1^{n-1} + p^n(\theta^n)^2, \qquad n = 1, 2, \ldots, N, \qquad (6\text{-}19)$$

$$x_1^0 = 0,$$

$$x_1^N \le P,$$

$$x_2^n = x_2^{n-1} + c^n(\theta^n + e^{\theta^n/4}), \qquad n = 1, 2, \ldots, N, \qquad (6\text{-}20)$$

$$x_2^0 = 0,$$

$$x_2^N \le C,$$

$$x_3^n = x_3^{n-1} + w^n \theta^n e^{\theta^n/4}, \qquad n = 1, 2, \ldots, N, \tag{6-21}$$

$$x_3^0 = 0,$$

$$x_3^N \leq W,$$

$$x_4^n = x_4^{n-1} + \ln\{1 - (1 - R^n)^{\theta^n}\}, \qquad n = 1, 2, \ldots, N, \tag{6-22}$$

$$x_4^0 = 0.$$

θ^n is constrained to positive integers only, that is,

$$\theta^n \geq 1.$$

The objective function to be maximized is

$$S = \sum_{n=1}^{N} \ln\left(1 - (1 - R^n)^{\theta^n}\right)$$

$$= x_4^N = \sum_{i=1}^{4} c_i x_i^N . \tag{6-23}$$

Therefore we obtain

$$c_i = 0, \qquad i = 1, 2, 3, \tag{6-24}$$

$$c_4 = 1.$$

The Hamiltonian and adjoint variables can be written as

$$H^n = \sum_{i=1}^{4} z_i^n x_i^n$$

$$= z_1^n\left(x_1^{n-1} + p^n(\theta^n)^2\right) + z_2^n\left(x_2^{n-1} + c^n(\theta^n + e^{\theta^n/4})\right)$$

$$+ z_3^n\left(x_3^{n-1} + w^n\theta^n e^{\theta^n/4}\right) + z_4^n\left(x_4^{n-1} + \ln(1 - (1 - R^n)^{\theta^n})\right), \qquad (6\text{-}25)$$

$n = 1, 2, 3, \ldots, N,$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n, \qquad (6\text{-}26)$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n, \qquad (6\text{-}27)$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_3^n, \qquad (6\text{-}28)$$

$$z_4^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_4^n, \qquad (6\text{-}29)$$

$$z_4^N = c_4 = 1. \qquad (6\text{-}29a)$$

From equations (6-29) and (6-29a), we obtain

$$z_4^n = 1, \qquad n = 1, 2, \ldots, N.$$

Hence equation (6-25) is reduced to

$$H^n = z_1^n\left(x_1^{n-1} + p^n(\theta^n)^2\right) + z_2^n\left(x_2^{n-1} + c^n(\theta^n + e^{\theta^n/4})\right)$$

$$+ z_3^n\left(x_3^{n-1} + w^n\theta^n e^{\theta^n/4}\right) + x_4^{n-1} + \ell n\left(1 - (1 - R^n)^{\theta^n}\right) . \qquad (6\text{-}30)$$

Differentiating equation (6-30) with respect to θ^n and equating to zero, we obtain

$$\frac{\partial H^n}{\partial \theta^n} = 0 = 2z_1^n p^n \theta^n + z_2^n c^n \left(1 + \frac{1}{4} e^{\theta^n/4}\right)$$

$$+ z_3^n w^n\left(e^{\theta^n/4} + \frac{1}{4}\,\theta^n e^{\theta^n/4}\right) + \frac{-(1-R^n)^{\theta^n}\,\ell n(1-R^n)}{1-(1-R^n)^{\theta^n}} . \qquad (6\text{-}31)$$

Let $U^n = 1 - R^n$.

Now if the first constraint is active and the rest of them are free, we obtain from equation (6-24) the following relations.

$$z_i^N = c_i = 0, \qquad i = 2, 3,$$

which gives

$$z_i^n = 0, \qquad n = 1, 2, \ldots, N;$$

$$i = 2, 3.$$

Therefore equation (6-31) may be written as

$$2z_1^n p^n \theta^n = \frac{(U^n)^{\theta^n}\,\ell n U^n}{1-(U^n)^{\theta^n}} . \qquad (6\text{-}32)$$

Solving equation (6-32) for z_1^n yields

$$z_1^n = \frac{1}{2p^n\theta^n} \frac{(U^n)^{\theta^n} \ln U^n}{1-(U^n)^{\theta^n}} \quad . \qquad (6\text{-}33)$$

Rearranging the terms in equation (6-32), we obtain

$$\theta^n = (U^n)^{\theta^n}\left(\theta^n + \frac{\ln U^n}{2z_1^n p^n}\right) \quad . \qquad (6\text{-}34)$$

Let $A^n = \dfrac{\ln U^n}{2z_1^n p^n}$

Therefore equation (6-34) becomes

$$\theta^n = (U^n)^{\theta^n}\left(\theta^n + A^n\right),$$

or

$$f(\theta^n) = \theta^n - (U^n)^{\theta^n}\left(\theta^n + A^n\right) \quad . \qquad (6\text{-}35)$$

The equation can be solved as explained in example 4, by Newton's method.

If the second constraint is active and the rest of them are free we obtain following relations from equation (6-24)

$$z_i^N = c_i = 0, \qquad i = 1, 3,$$

which gives

$$z_i^n = 0, \qquad n = 1, 2, \ldots, N$$

$$i = 1, 3.$$

Therefore equation (6-31) may be written as

$$z_2^n c^n \left(1 + \frac{1}{4} e^{\theta^n/4}\right) = \frac{(U^n)^{\theta^n} \ln U^n}{1 - (U^n)^{\theta^n}} \quad . \tag{6-36}$$

solving equation (6-36) for z_2^n yields

$$z_2^n = \frac{1}{c^n\left(1 + \frac{1}{4} e^{\theta^n/4}\right)} \left(\frac{(U^n)^{\theta^n} \ln U^n}{1 - (U^n)^{\theta^n}}\right) \quad . \tag{6-37}$$

Rearranging the terms in equation (6-36), we obtain

$$f(\theta^n) = \left(1 + \frac{1}{4} e^{\theta^n/4}\right) - (U^n)^{\theta^n} \left(\left(1 + \frac{1}{4} e^{\theta^n/4}\right) + \frac{\ln U^n}{z_2^n c^n}\right) \quad . \tag{6-38}$$

The equation is well behaved and therefore Newton's method can be applied to solve it.

Lastly if the third constraint is active and rest of them are free we obtain from equation (6-24) the following relations

$$z_i^N = c_i = 0, \qquad i = 1, 2.$$

This gives

$$z_i^n = 0, \qquad n = 1, 2, \ldots, N, \quad i = 1, 2.$$

Therefore equation (6-31) may be written as

$$z_3^n w^n \left(e^{\theta^n/4} + \frac{1}{4}\,\theta^n e^{\theta^n/4}\right) = \frac{(U^n)^{\theta^n} \ln U^n}{1 - (U^n)^{\theta^n}} \quad . \tag{6-39}$$

Solving equation (6-39) for z_3^n yields

$$z_3^n = \frac{1}{w^n\left(e^{\theta^n/4} + \frac{1}{4}\,\theta^n e^{\theta^n/4}\right)} \left(\frac{(U^n)^{\theta^n} \ln U^n}{1 - (U^n)^{\theta^n}}\right) \quad . \tag{6-40}$$

Rearranging the terms in equation (6-39), we obtain

$$f(\theta^n) = e^{\theta^n/4}\left(1 + \frac{1}{4}\,\theta^n\right) - (U^n)^{\theta^n}\left(e^{\theta^n/4}\left(1 + \frac{1}{4}\,\theta^n\right) + \frac{\ln U^n}{z_3^n w^n}\right\} \quad . \tag{6-41}$$

This is again a well behaved equation and can be solved by Newton's method.

The computer program and the flow diagram for solving this problem by the method explained in section 5 are given in Appendix F. The computer program is for an N-stage system.

NUMERICAL EXAMPLES:

Three stage and five stage problems are solved. The constants assigned for the three stage problem are given in Table 12. The optimum redundancy obtained is as follows (see Appendix F, Table F3).

$\theta^1 = 2.4000,$

$\theta^2 = 1.8666,$

$\theta^3 = 3.1465.$

Since θ^n, n = 1, 2, 3 should be positive integers, we obtain

Table 12. Constants assigned for 3 stage problem.

n	R^n	p^n	P	c^n	C	w^n	W
1	.80	1		7		7	
2	.90	3	70	5	110	8	100
3	.65	4		9		6	

$$\theta^1 = 2,$$

$$\theta^2 = 2,$$

$$\theta^3 = 3.$$

The number of redundant elements at each stage can be obtain by subtracting one from each of the above figure.

From result we find that the total of the product of volume and weight is 52 with 18 units of slacks, the cost of the system is 89.84 with 20.16 units of slacks and the weight of the system is 87.57 with 12.43 units of slacks. This policy results in a total system reliability of 0.9097.

The constants assigned for five stage problem are given in Table 13. The optimum redundancy obtained is as follows, (see Appendix F, Table F4).

$$\theta^1 = 2.6000,$$

$$\theta^2 = 2.2816,$$

$$\theta^3 = 2.0075,$$

$$\theta^4 = 2.6882,$$

$$\theta^5 = 3.3981.$$

Since θ^n, $n = 1, 2, \ldots, 5$ should be positive integers, we obtain

$$\theta^1 = 3,$$

$$\theta^2 = 2,$$

$$\theta^3 = 2,$$

$$\theta^4 = 3,$$

$$\theta^5 = 3.$$

Table 13. Constants assigned for 5 stage problem.

n	R^n	p^n	P	c^n	C	w^n	W
1	.80	1		7		7	
2	.85	2		7		8	
3	.90	3	110	9	175	6	200
4	.65	4		9		6	
5	.75	2		4		9	

The number of redundant elements at each stage can be obtained by subtracting one from each of the above figures.

From the result we find that the total of the product of weight and volume is 83 with a slack of 27 units, the cost of the system is 146.12 with a slack of 26.88 units and the weight of the system is 192.48 with a slack of 7.52 units. This policy results in a system reliability of 0.9045.

7. PROPOSALS FOR FUTURE STUDIES

PROPOSAL 1.

In our discussion in problems 1 to 6, we have assumed θ^n to be continuous variables, while in fact they are positive integers only. Rounding off to the nearest integer may or may not give the true optimum value. To avoid this ambiguity, in this section we will see how zero one integer programming can be applied for round off. Suppose we need to find an integer solution to the following separable programming problem.

Optimize

$$F = \sum_{n=1}^{N} f^n(\theta^n), \tag{7-1}$$

subject to

$$\sum_{n=1}^{N} g_i^n(\theta^n) \leq b_i, \quad i = 1, 2, \ldots, r. \tag{7-2}$$

Let the integer part of the optimum decision obtained by solution of above problem be given by $(\bar{\theta}^n)_1$, $n = 1, 2, \ldots, N$. Let $\bar{\theta}^n$ denote the true optimum integer solution. Then

$$\bar{\theta}^n = (\bar{\theta}^n)_1 + (\theta^n)_2 ,$$

where

$$(\theta^n)_2 = 0 \text{ or } 1.$$

The new problem can now be stated as,

Optimize

$$F = \sum_{n=1}^{N} f^n\{(\bar{\theta}^n)_1\} + \sum_{n=1}^{N} \left(f^n\{(\bar{\theta}^n)_1 + 1\} - f^n\{(\bar{\theta}^n)_1\}\right) (\theta^n)_2 ,$$

subject to

$$\sum_{n=1}^{N} g_i^n\{(\bar{\theta}^n)_1\} + \sum_{n=1}^{N} \left(g_i^n\{(\bar{\theta}^n)_1 + 1\} - g_i^n\{(\bar{\theta}^n)_1\}\right)(\theta^n)_2 \leq b_i ,$$

$$i = 1, 2, \ldots , r, \qquad (7\text{-}4)$$

and

$$(\theta^n)_2 = 0 \text{ or } 1.$$

Now let

$$\sum_{n=1}^{N} f^n\{(\bar{\theta}^n)_1\} = \bar{F}_1 ,$$

and

$$\sum_{n=1}^{N} g_i^n\{(\bar{\theta}^n)_1\} = \bar{b}_{1i}, \qquad i = 1, 2, \ldots , r.$$

$\bar{F}_1$ and $\bar{b}_{1i}$ being constants, the above formulation may be simplified to,

Optimize

$$F_2 = \sum_{n=1}^{N} \left(f^n\{(\bar{\theta}^n)_1 + 1\} - f^n\{(\bar{\theta}^n)_1\}\right)(\theta^n)_2 , \qquad (7\text{-}5)$$

subject to

$$\sum_{n=1}^{N} \left(g_i^n\{(\bar{\theta}^n)_1 + 1\} - g_i^n\{(\bar{\theta}^n)_1\}\right)(\theta^n)_2 \leq b_{2i}, \quad i = 1, 2, \ldots, r, \tag{7-6}$$

and

$$(\theta^n)_2 = 0 \text{ or } 1.$$

Where

$$F_2 = F - \bar{F}_1,$$

and

$$b_{2i} = b_i - \bar{b}_{1i} .$$

Both the functions (7-5) and (7-6) are linear, and the decision variable is restricted to zero or one. Thus this problem may be solved by zero one integer programming.

PROPOSAL 2.

In the application of the discrete maximum principle to problems with separable functions as given in section 5, many questions need to be answered.

(i) How to obtain the initial estimate for θ^1,

(ii) If more than one constraint is violated before the constraint considered to be active reaches its limit, then which constraint should be selected as the active constraint so that the process of convergence to optimum is fastest,

(iii) With assumed value for θ^1, if no constraint is violated, what is the best increase in assumed value of θ^1, for the rapid convergence to the optimum.

PROPOSAL 3.

In this report, we have considered parallel redundancy only. It is possible to increase the system reliability by stand-by redundancy. In parallel redundancy all redundant components are working simultaneously but in standby redundancy, a redundant component will be switched into operation when the original component fails.

PROPOSAL 4.

Optimal choice of Design: In problems solved in this report, it is assumed that at any stage there is only one possible design. There may be a number of designs that are possible at any stage. What is meant by number of designs is that there may be more than one type of the component available performing the same operation but with different reliability, weight, cost and volume. Under such a condition, the problem is not only to select the optimum number of elements at every stage but also the optimal design.

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Appendix A. Computer Flow Chart, Symbol Table, Computer Program and Results for Example 1.

Fig. A1. COMPUTOR FLOW CHART

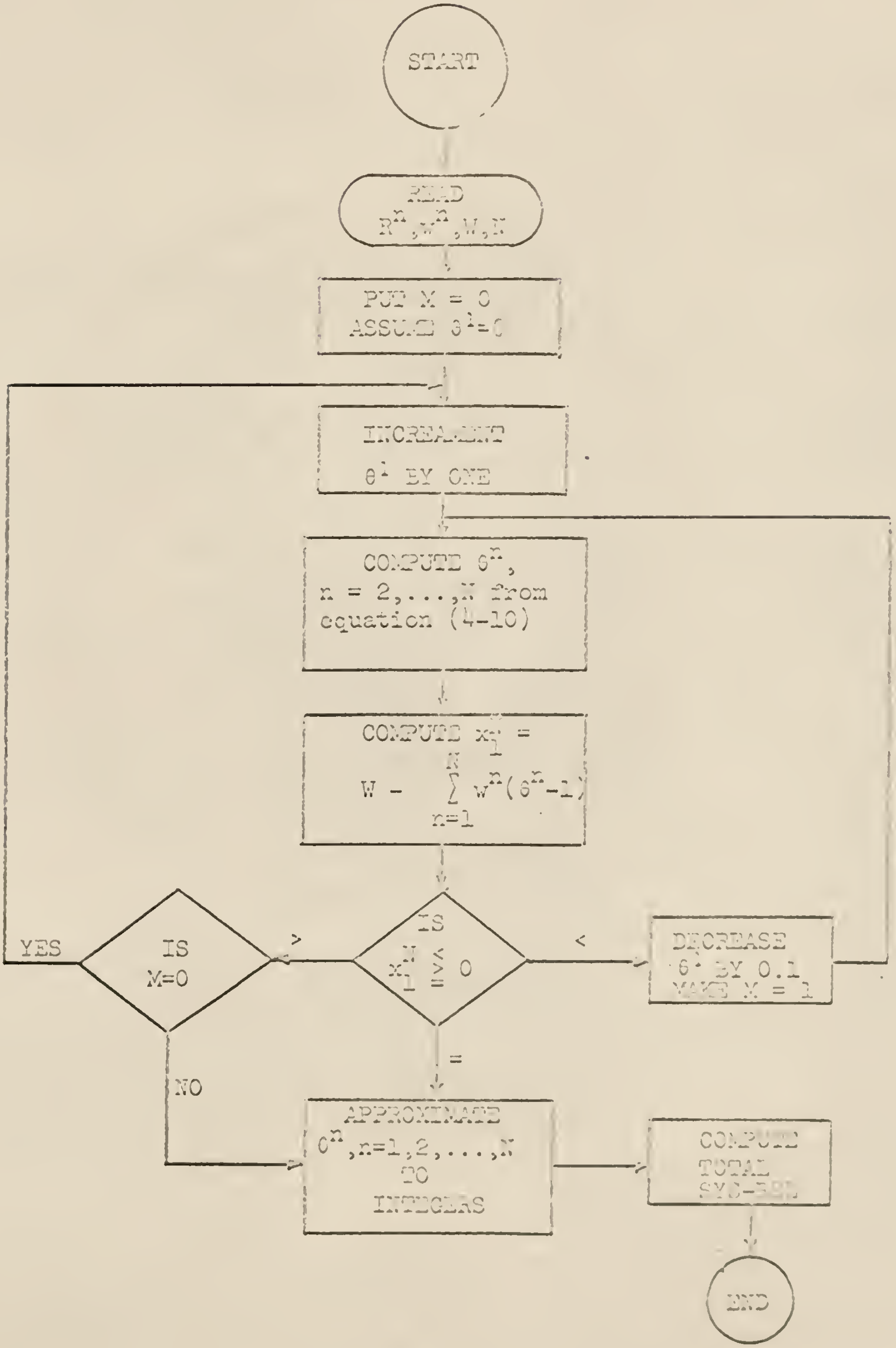

Table A1. Symbol table for computer program

Program Symbol	Mathematical Symbol	Explanation
REL	R_s	Total system reliability.
WT		Total weight in redundant elements.
PATT	$x^0 = W$	Total allowable weight in redundant elements.
X(N)	x_1^N	Weight left after allocating N stages.
UR(I)	$1 - R^n$	Unreliability.
IRED(I)	$\theta^n - 1$	Redundant elements at n^{th} stage.
THETA(I)	θ^n	Number of elements at n^{th} stage.
M		Dummy number to keep track in variation of θ^1.
R(I)	R^n	Reliability of one unit in n^{th} stage.
W(I)	W^n	Weight of a unit in n^{th} stage
ITHTA		Integer portion of θ(I).

TABLE A2. COMPUTER PROGRAM FOR EXAMPLE 1.

```
C  C  MAXIMIZATION OF SYS.REL.SUBJECT TO WT CONSTRAINTS
      DIMENSION R(10),W(10),X(10),THETA(10),UR(10),IRED(10)
    1 FORMAT(I5,F10.2)
    3 FORMAT(F6.4,F5.2)
  104 FORMAT(28HDATA FOR THE PROBLEM FOLLOWS)
  105 FORMAT(13HNO OF STAGES=I3)
  106 FORMAT(37HTOTAL ALLOWABLE WT IN REDUNDANT ELE.=F6.2)
  100 FORMAT(1H )
  107 FORMAT(8HSTAGE NO5X11HRELIABILITY5X11HWT/RED.ELE.)
  108 FORMAT(3XI3,9XF6.4,12XF5.2)
  109 FORMAT(28HOPTIMUM POLICY IS AS FOLLOWS)
  101 FORMAT(6HSTAGE ,I3,3X21HNO OF REDUNDANT ELE.=,F7.4,3X,3HUSE,I5)
  103 FORMAT(9HTOTAL WT=,2XF9.2,3X9HSYS.REL.=,2XF6.4)
      REL=1.
      WT=0
      M=0
      THETA(1)=0
      READ1,N,PATT
      DO 2 I=1,N
      READ 3,R(I),W(I)
    2 CONTINUE
      PUNCH 104
      PUNCH 105,N
      PUNCH 106,PATT
      PUNCH 100
      PUNCH 107
   40 THETA(1)=THETA(1)+1.
   31 X(1)=PATT-W(1)*(THETA(1)-1.)
      K=N-1
      UR(1)=1.-R(1)
      DO 13 I=1,K
      UR(I+1)=1.-R(I+1)
      Q=1.+(W(I)/W(I+1))*(LOG(UR(I+1))/LOG(UR(I)))
      P=1./UR(I)**THETA(I)-1.
      THETA(I+1)=-(LOG(P*(Q-1.)+1.)/LOG(UR(I+1)))
      X(I+1)=X(I)-W(I+1)*(THETA(I+1)-1.)
   13 CONTINUE
      IF(X(N)) 16,19,15
   15 IF(M) 40,40,19
   16 THETA(1)=THETA(1)-.1
      M=M+1
      GO TO 31
   19 DO 71 I=1,N
   71 PUNCH 108,I,R(I),W(I)
      PUNCH 100
      PUNCH 109
      PUNCH 100
      DO 61 I=1,N
      ITHTA=THETA(I)
      DUMMY=ITHTA
      DIFF=THETA(I)-DUMMY
      IF(DIFF-.5) 52,53,53
   52 IRED(I)=DUMMY-1.
      GO TO 54
   53 IRED(I)=DUMMY
   54 IF(THETA(I)) 56,56,51
   56 THETA(I)=1.
   51 REDNT=THETA(I)-1.
      PUNCH 101,I,REDNT,IRED(I)
      RED=IRED(I)
      WT=WT+W(I)*RED
   61 REL=REL*(1.-(1.-R(I))**(RED+1.))
      PUNCH 100
      PUNCH 103,WT,REL
      END
```

SAMPLE DATA CARDS *****

```
    N     PATT
    5     104.0

  R     W
 .90    8.
 .75    9.
 .65    6.
 .80    7.
 .85    8.
```

TABLE A3. COMPUTER OUTPUT OF FIVE STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  5
TOTAL ALLOWABLE WT IN REDUNDANT ELE.=104.00

STAGE NO      RELIABILITY      WT/RED.ELE.
     1           .9000              8.00
     2           .7500              9.00
     3           .6500              6.00
     4           .8000              7.00
     5           .8500              8.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE    1   NO OF REDUNDANT ELE.= 1.6000   USE     2
STAGE    2   NO OF REDUNDANT ELE.= 2.8691   USE     3
STAGE    3   NO OF REDUNDANT ELE.= 4.2300   USE     4
STAGE    4   NO OF REDUNDANT ELE.= 2.5806   USE     3
STAGE    5   NO OF REDUNDANT ELE.= 2.0539   USE     2

TOTAL WT=      104.00    SYS.REL.=    .9850
```

TABLE A4. COMPUTER OUTPUT OF TEN STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES= 10
TOTAL ALLOWABLE WT IN REDUNDANT ELE.=200.00

STAGE NO       RELIABILITY      WT/RED.ELE.
     1            .7800            17.00
     2            .8600            19.00
     3            .8800            25.00
     4            .8000             8.00
     5            .7000             9.00
     6            .9000            16.00
     7            .6500            21.00
     8            .9400            11.00
     9            .7900            18.00
    10            .8600            18.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.4000   USE    1
STAGE   2   NO OF REDUNDANT ELE.=  .9227   USE    1
STAGE   3   NO OF REDUNDANT ELE.=  .6914   USE    1
STAGE   4   NO OF REDUNDANT ELE.= 1.7549   USE    2
STAGE   5   NO OF REDUNDANT ELE.= 2.4449   USE    2
STAGE   6   NO OF REDUNDANT ELE.=  .7822   USE    1
STAGE   7   NO OF REDUNDANT ELE.= 1.9308   USE    2
STAGE   8   NO OF REDUNDANT ELE.=  .6604   USE    1
STAGE   9   NO OF REDUNDANT ELE.= 1.3117   USE    1
STAGE  10   NO OF REDUNDANT ELE.=  .9496   USE    1

TOTAL WT=     198.00   SYS.REL.=   .7853
```

Appendix B. Computer Flow Chart, Symbol Table, Computer Program and Results for Example 2.

Fig. 11. COMPUTER FLOW DIAGRAM

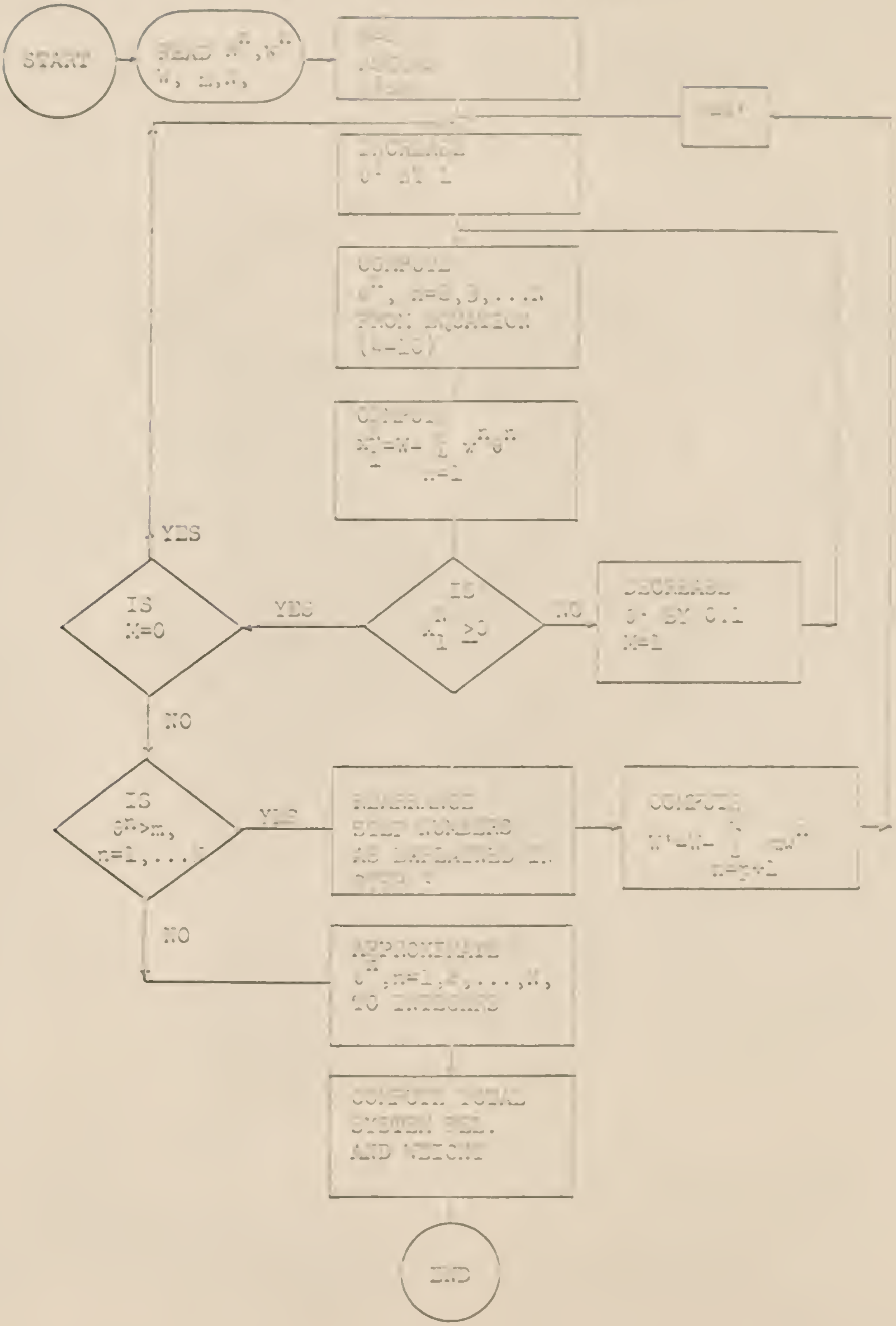

Table B1. Symbol table for computer program.

Program Symbol	Mathematical Symbol	Explanation
REL	R_s	Total system reliability.
WT		Total weight of the system.
PATT	$x_1^0 = W$	Total allowable weight in redundant elements.
M1	$m - 1$	Maximum number of redundant elements allowed per stage.
X(N)	x_1^N	Weight left over after allocating N stages.
UR(I)	$1 - R^n$	Unreliability of one element at nth stage.
IRED(I)	$\theta^n - 1$	Redundant elements at nth stage.
THETA(I)	θ^n	Number of elements at nth stage.
R(I)	R^n	Reliability of one element at nth stage.
W(I)	w^n	Weight per unit at nth stage.
ITHTA		Integer portion of θ^n.
M		Dummy variable to keep track in variation of θ^1.
N1	p	Number of stages for which θ^n is less than m.

TABLE B2.   COMPUTER PROGRAM FOR EXAMPLE 2.

```
C  C   MAXIMIZATION OF SYS.REL.SUBJECT TO WT CONSTRAINTS
       DIMENSION R(10),W(10),X(10),THETA(10),UR(10),IRED(10)
     1 FORMAT(I5,F10.2)
     3 FORMAT(F6.4,F5.2)
     4 FORMAT(I3)
   110 FORMAT(45HMAXIMUM REDUNDANT ELEMENTS ALLOWED PER STAGE=,I4)
   104 FORMAT(28HDATA FOR THE PROBLEM FOLLOWS)
   105 FORMAT(13HNO OF STAGES=I3)
   106 FORMAT(37HTOTAL ALLOWABLE WT IN REDUNDANT ELE.=F6.2)
   100 FORMAT(1H )
   107 FORMAT(8HSTAGE NO5X11HRELIABILITY5X11HWT/RED.ELE.)
   108 FORMAT(3XI3,9XF6.4,12XF5.2)
   109 FORMAT(28HOPTIMUM POLICY IS AS FOLLOWS)
   101 FORMAT(6HSTAGE ,I3,3X21HNO OF REDUNDANT ELE.=,F7.4,3X,3HUSE,I5)
   103 FORMAT(9HTOTAL WT=,2XF9.2,3X9HSYS.REL.=,2XF6.4)
       L=1
       REL=1.
       WT=0
       M=0
       THETA(1)=0
       READ1,N,PATT
       READ4,M1
       C=M1
       C=C+.5
       DO 2 I=1,N
       READ 3,R(I),W(I)
     2 CONTINUE
       PUNCH 104
       PUNCH 105,N
       PUNCH 106,PATT
       PUNCH 110,M1
       PUNCH 100
       PUNCH 107
    40 THETA(1)=THETA(1)+1.
    31 X(1)=PATT-W(1)*(THETA(1)-1.)
       K=N-L
       IF(N-L) 33,33,32
    33 THETA(1)=PATT/W(1)+1.
       GO TO 25
    32 UR(1)=1.-R(1)
       DO 13 I=1,K
       UR(I+1)=1.-R(I+1)
       Q=(W(I)/W(I+1))*(LOG(UR(I+1))/LOG(UR(I)))
       P=1./UR(I)**THETA(I)-1.
       THETA(I+1)=-(LOG(P*Q+1.)/LOG(UR(I+1)))
       X(I+1)=X(I)-W(I+1)*(THETA(I+1)-1.)
    13 CONTINUE
```

```
      N1=N-L+1
      IF(X(N1)) 16,19,15
   15 IF(M) 40,40,19
   16 THETA(1)=THETA(1)-.1
      M=M+1
      GO TO 31
   19 I1=1
   23 L1=N-L+1
      DO 21 I=I1,L1
      IF(THETA(I)-C) 21,22,22
   22 IF(I-L1)80,35,35
   80 WET=W(I)
      RE=R(I)
      W(I)=W(L1)
      R(I)=R(L1)
      W(L1)=WET
      R(L1)=RE
      THETA(I)=THETA(L1)
   35 THETA(L1)=C+.5
      PATT=PATT-W(L1)*(THETA(L1)-1.)
      L=L+1
      M=0
      IF(I-L1) 10,21,21
   10 GO TO 26
   21 CONTINUE
      I=0
   26 IF(I) 27,27,24
   24 I1=I
      I=0
      GO TO 23
   27 IF(M) 40,40,25
   25 DO 71 I=1,N
   71 PUNCH 108,I,R(I),W(I)
      PUNCH 100
      PUNCH 109
      PUNCH 100
      DO 61 I=1,N
      ITHTA=THETA(I)
      DUMMY=ITHTA
      DIFF=THETA(I)-DUMMY
      IF(DIFF-.5) 52,53,53
   52 IRED(I)=DUMMY-1.
      GO TO 54
   53 IRED(I)=DUMMY
   54 IF(THETA(I)) 56,56,51
   56 THETA(I)=1.
   51 THETA(I)=THETA(I)-1.
      PUNCH 101,I,THETA(I),IRED(I)
      RED=IRED(I)
      WT=WT+W(I)*RED
   61 REL=REL*(1.-(1.-R(I))**(RED+1.))
      PUNCH 100
      PUNCH 103,WT,REL
      END
```

```
  SAMPLE DATA CARDS  *****


     N    PATT
     3     75.
  4               CONSTRAINT ON REDUNDANT ELEMENTS
  R      W
 .95    8.
 .80    9.
 .65    7.

             *************            **************
```

TABLE B3.   COMPUTER OUTPUT FOR THREE STAGE PROBLEM
WITH THETA RESTRICTED TO MAX. 5

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  3
TOTAL ALLOWABLE WT IN REDUNDANT ELE.= 75.00
MAXIMUM REDUNDANT ELEMENTS ALLOWED PER STAGE=    4

STAGE NO      RELIABILITY      WT/RED.ELE.
     1            .9500             8.00
     2            .8000             9.00
     3            .6500             7.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.3750   USE    1
STAGE   2   NO OF REDUNDANT ELE.= 4.0000   USE    4
STAGE   3   NO OF REDUNDANT ELE.= 4.0000   USE    4

TOTAL WT=       72.00   SYS.REL.=   .9919
```

TABLE B4. COMPUTER OUTPUT FOR THREE STAGE PROBLEM WITH THETA UNCONSTRAINED

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  3
TOTAL ALLOWABLE WT IN REDUNDANT ELE.= 75.00
MAXIMUM REDUNDANT ELEMENTS ALLOWED PER STAGE=  50

STAGE NO      RELIABILITY      WT/RED.ELE.
    1             .9500             8.00
    2             .8000             9.00
    3             .6500             7.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.4000   USE    1
STAGE   2   NO OF REDUNDANT ELE.= 3.0085   USE    3
STAGE   3   NO OF REDUNDANT ELE.= 4.9780   USE    5

TOTAL WT=       70.00   SYS.REL.=   .9941
```

```
   TABLE B5.   COMPUTER OUTPUT FOR TEN STAGE PROBLEM
               WITH THETA CONSTRAINED TO MAX. 5

DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES= 10
TOTAL ALLOWABLE WT IN REDUNDANT ELE.=475.00
MAXIMUM REDUNDANT ELEMENTS ALLOWED PER STAGE=   4

STAGE NO       RELIABILITY      WT/RED.ELE.
     1             .7800             17.00
     2             .8600             19.00
     3             .8800             25.00
     4             .8000             15.00
     5             .8600             18.00
     6             .9000             16.00
     7             .7900             18.00
     8             .9400             11.00
     9             .6500             21.00
    10             .7000             14.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO  OF REDUNDANT ELE.= 3.1000   USE     3
STAGE   2   NO  OF REDUNDANT ELE.= 2.2336   USE     2
STAGE   3   NO  OF REDUNDANT ELE.= 1.9049   USE     2
STAGE   4   NO  OF REDUNDANT ELE.= 2.9727   USE     3
STAGE   5   NO  OF REDUNDANT ELE.= 2.2511   USE     2
STAGE   6   NO  OF REDUNDANT ELE.= 1.9041   USE     2
STAGE   7   NO  OF REDUNDANT ELE.= 2.9606   USE     3
STAGE   8   NO  OF REDUNDANT ELE.= 1.5810   USE     2
STAGE   9   NO  OF REDUNDANT ELE.= 4.0000   USE     4
STAGE  10   NO  OF REDUNDANT ELE.= 4.0000   USE     4

TOTAL WT=      468.00   SYS.REL.=    .9782
```

TABLE B6. COMPUTER OUTPUT FOR TEN STAGE PROBLEM
WITH THETA UNCONSTRAINED

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES= 10
TOTAL ALLOWABLE WT IN REDUNDANT ELE.=475.00
MAXIMUM REDUNDANT ELEMENTS ALLOWED PER STAGE=  50

STAGE NO      RELIABILITY      WT/RED.ELE.
     1            .7800            17.00
     2            .8600            19.00
     3            .8800            25.00
     4            .7000            14.00
     5            .8000            15.00
     6            .9000            16.00
     7            .6500            21.00
     8            .9400            11.00
     9            .7900            18.00
    10            .8600            18.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 3.0000   USE    3
STAGE   2   NO OF REDUNDANT ELE.= 2.1566   USE    2
STAGE   3   NO OF REDUNDANT ELE.= 1.8335   USE    2
STAGE   4   NO OF REDUNDANT ELE.= 4.0014   USE    4
STAGE   5   NO OF REDUNDANT ELE.= 2.8756   USE    3
STAGE   6   NO OF REDUNDANT ELE.= 1.8383   USE    2
STAGE   7   NO OF REDUNDANT ELE.= 4.2207   USE    4
STAGE   8   NO OF REDUNDANT ELE.= 1.5271   USE    2
STAGE   9   NO OF REDUNDANT ELE.= 2.8636   USE    3
STAGE  10   NO OF REDUNDANT ELE.= 2.1840   USE    2

TOTAL WT=     468.00   SYS.REL.=   .9782
```

Appendix C. Computer Flow Chart, Symbol Table, Computer Program and Results For Example 3.

Fig. C1. Computer Flow Chart

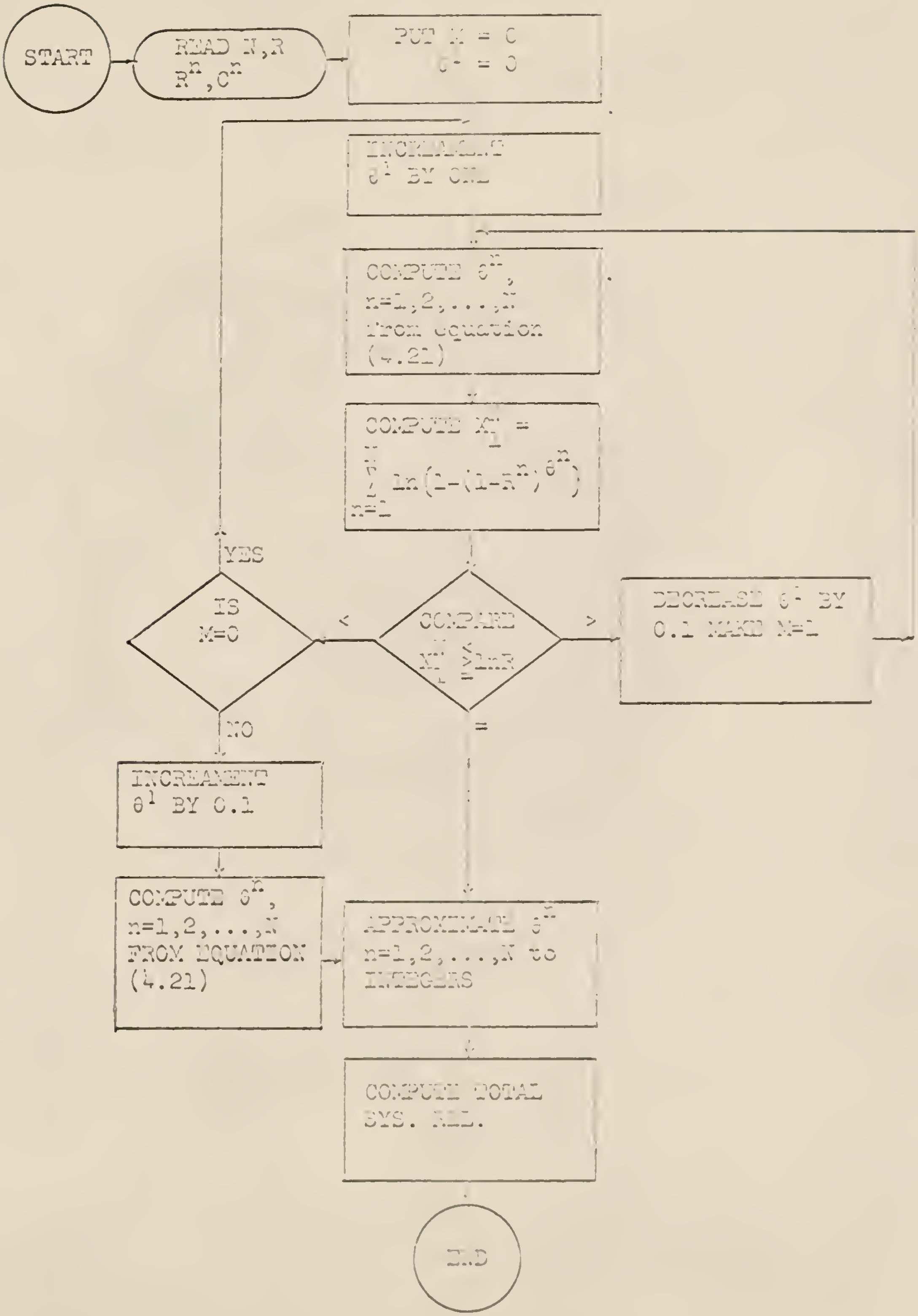

Table C1. Symbol table for computer program.

Program Symbol	Mathematical Symbol	Explanation
COST		Total cost of the system.
THETA(I)	θ^n	Number of elements at nth stage.
RELB	R	Minimum level of reliability to be achieved.
R(I)	R^n	Reliability of nth stage.
C(I)	c^n	Cost of one element at nth stage.
REL	R_s	Total system reliability.
UR(I)	$1 - R^n$	Unreliability of one element at nth stage.
ITHTA		Integer portion of θ^n.
IRED(I)	$\theta^n - 1.$	Number of redundant elements at nth stage.

TABLE C2. COMPUTER PROGRAM FOR EXAMPLE 3.

```
C  C  MINIMIZATION OF THE COST OF REDUNDENT ELEMENTS WHILE ACHIEVING
C     A MINIMUM LEVEL OF RELIABILITY
      DIMENSION R(10),C(10),THETA(10),UR(10),IRED(10)
    1 FORMAT(I3,F10.4)
    3 FORMAT(F6.4,F5.2)
  103 FORMAT(24HCOST OF REDUNDANT UNITS=2XF10.2,3X9HSYS.REL.=2XF6.4)
  104 FORMAT(28HDATA FOR THE PROBLEM FOLLOWS)
  105 FORMAT(13HNO OF STAGES=I3)
  106 FORMAT(26HMINIMUM RELIABILITY LEVEL=F6.4)
  100 FORMAT(1H )
  107 FORMAT(8HSTAGE NO5X11HRELIABILITY5X13HCOST/RED.ELE.)
  108 FORMAT(3XI3,9XF6.4,12XF5.2)
  109 FORMAT(28HOPTIMUM POLICY IS AS FOLLOWS)
  101 FORMAT(6HSTAGE ,I3,3X21HNO OF REDUNDANT ELE.=,F7.4,3X,3HUSE,I5)
      L=1
      COST=0
      M1=0
      M=0
      THETA(1)=0
      READ 1,N,RELB
      DO 2 I=1,N
      READ 3,R(I),C(I)
    2 CONTINUE
      PUNCH 104
      PUNCH 105,N
      PUNCH 106,RELB
      PUNCH 100
      PUNCH 107
   40 THETA(1)=THETA(1)+1.
   31 REL=1.-(1.-R(1))**THETA(1)
      K=N-L
      UR(1)=1.-R(1)
   32 DO 13 I=1,K
      UR(I+1)=1.-R(I+1)
      Q=1.+(C(I)/C(I+1))*(LOG(UR(I+1))/LOG(UR(I)))
      P=1./UR(I)**THETA(I)-1.
      THETA(I+1)=-(LOG(P*(Q-1.)+1.)/LOG(UR(I+1)))
      REL=REL*(1.-(1.-R(I+1))**THETA(I+1))
   13 CONTINUE
      IF(M1) 4,4,5
    4 IF(RELB-REL) 16,19,15
   15 IF(M) 40,40,19
   16 THETA(1)=THETA(1)-.1
      M=M+1
      GO TO 31
   19 THETA(1)=THETA(1)+.1
      M1=1
      GO TO 31
    5 DO 71 I=1,N
   71 PUNCH 108,I,R(I),C(I)
      REL=1.
      PUNCH 100
      PUNCH 109
      PUNCH 100
      DO 61 I=1,N
      ITHTA=THETA(I)
      DUMMY=ITHTA
      DIFF=THETA(I)-DUMMY
      IF(DIFF-.5) 52,53,53
   52 IRED(I)=DUMMY-1.
      GO TO 54
   53 IRED(I)=DUMMY
   54 IF(THETA(I)) 56,56,51
   56 THETA(I)=1.
   51 THETA(I)=THETA(I)-1.
      PUNCH 101,I,THETA(I),IRED(I)
      RED=IRED(I)
      COST=COST+C(I)*RED
   61 REL=REL*(1.-(1.-R(I))**(RED+1.))
      PUNCH 100
      PUNCH 103,COST,REL
      END
```

```
 SAMPLE DATA CARDS  *****

   N  LEVEL OF R

   R      C
   5     .90
.65   2.
.55   3.
.70   6.
.75   7.
.60   4.
```

```
  TABLE C3.   COMPUTER OUTPUT FOR FIVE STAGE PROBLEM

DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  5
MINIMUM RELIABILITY LEVEL= .9000

STAGE NO     RELIABILITY     COST/RED.ELE.
     1          .6500           2.00
     2          .5500           3.00
     3          .7000           6.00
     4          .7500           7.00
     5          .6000           4.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 3.4000   USE    3
STAGE   2   NO OF REDUNDANT ELE.= 3.9463   USE    4
STAGE   3   NO OF REDUNDANT ELE.= 2.0511   USE    2
STAGE   4   NO OF REDUNDANT ELE.= 1.6406   USE    2
STAGE   5   NO OF REDUNDANT ELE.= 3.1501   USE    3

COST OF REDUNDANT UNITS=       56.00   SYS.REL.=   .9023
```

TABLE C4. COMPUTER OUTPUT FOR TEN STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES= 10
MINIMUM RELIABILITY LEVEL= .8500

STAGE NO      RELIABILITY      COST/RED.ELE.
     1           .7500              7.00
     2           .6000              4.00
     3           .9500              4.00
     4           .8000             10.00
     5           .8500              4.00
     6           .9000              8.00
     7           .8000              6.00
     8           .5500              3.00
     9           .7000              6.00
    10           .6500              2.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.8000   USE    2
STAGE   2   NO OF REDUNDANT ELE.= 3.3920   USE    3
STAGE   3   NO OF REDUNDANT ELE.=  .7346   USE    1
STAGE   4   NO OF REDUNDANT ELE.= 1.2859   USE    1
STAGE   5   NO OF REDUNDANT ELE.= 1.5000   USE    2
STAGE   6   NO OF REDUNDANT ELE.=  .8453   USE    1
STAGE   7   NO OF REDUNDANT ELE.= 1.5969   USE    2
STAGE   8   NO OF REDUNDANT ELE.= 4.2247   USE    4
STAGE   9   NO OF REDUNDANT ELE.= 2.2347   USE    2
STAGE  10   NO OF REDUNDANT ELE.= 3.6136   USE    4

COST OF REDUNDANT UNITS=       100.00   SYS.REL.=   .8541
```

Appendix D. Computer Flow Chart, Symbol Table, Computer Program and Results for Example 4.

Fig. 31. COMPUTER FLOW CHART

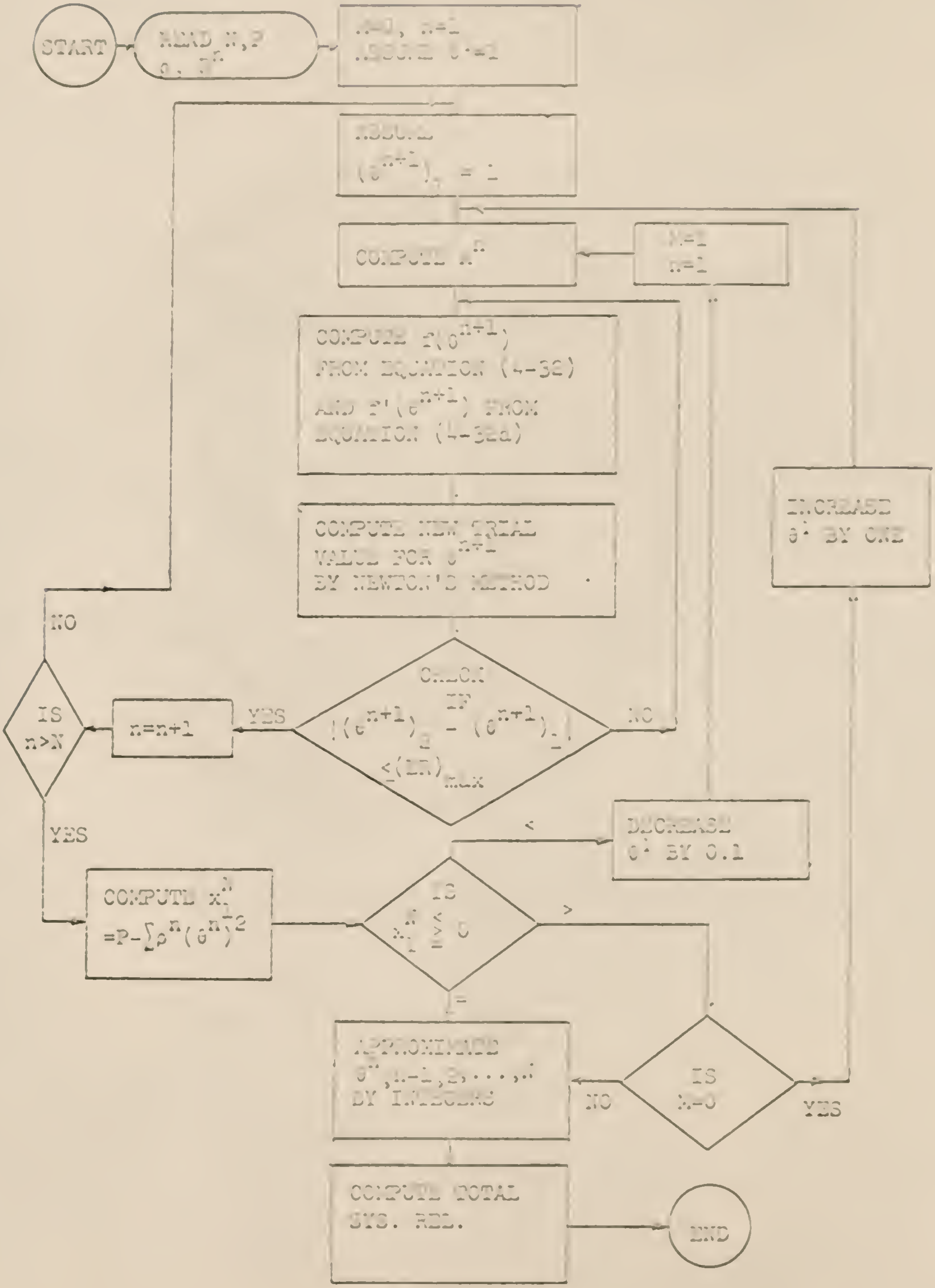

Table D1. Symbol table for computer program.

Program Symbol	Mathematical Symbol	Explanation
REL	R_s	Total system reliability
PR		Total constraint limit reached
THETA(I)	θ^n	Number of elements at nth stage
PATT	P	Constraint limit allowed
R(I)	R^n	Reliability per element at nth stage
P(I)	p^n	Constraint constant for nth stage.
X(I)	x_1^n	Constraint limit left over after allocating n stages
U(I)	$1 - R^n$	Unreliability per element at nth stage
A	A^n	Constant for nth stage as given in equation (4-32)
THET	$f(\theta^{n+1})$	Value of the expression given by equation (4-32)
THTA	$f'(\theta^{n+1})$	Value of the expression given by equation (4-32a)
ITHTA		Integer portion of θ^n
IRED(I)	θ^n-1	Number of redundant elements at nth stage

TABLE D2. COMPUTER PROGRAM FOR EXAMPLE 4

```
C     C   MAXIMIZATION OF SYSTEM RELIABILITY SUBJECT TO NON LINEAR CONSTRAI
          DIMENSION R(15),P(10),X(10),THETA(15),U(10),IRED(10)
        1 FORMAT(I3,F6.2)
        3 FORMAT(F6.4,F5.2)
      104 FORMAT(28HDATA FOR THE PROBLEM FOLLOWS)
      105 FORMAT(13HNO OF STAGES=I3)
      106 FORMAT(18HCONSTRAINT LIMIT =,F6.2)
      100 FORMAT(1H )
      107 FORMAT(8HSTAGE NO5X11HRELIABILITY10X4HP(I))
      108 FORMAT(3XI3,9XF6.4,12XF5.2)
      109 FORMAT(28HOPTIMUM POLICY IS AS FOLLOWS)
      101 FORMAT(6HSTAGE ,I3,3X21HNO OF REDUNDANT ELE.=,F7.4,3X,5HUSE,I5)
      103 FORMAT(21HSLACK IN CONSTRAINT =,F9.4,3X9HSYS.REL.=,2XF6.4)
          REL=1.
          PR=0
          M1=0
          THETA(1)=0
          READ 1,N,PATT
          DO 2 I=1,N
        2 READ 3,R(I),P(I)
          PUNCH 104
          PUNCH 105,N
          PUNCH 106,PATT
          PUNCH 100
          PUNCH 107
       40 THETA(1)=THETA(1)+1.
       39 X(1)=PATT-P(1)*THETA(1)**2.
          K=N-1
          DO 13 I=1,K
          U(I)=1.-R(I)
          U(I+1)=1.-R(I+1)
          A1=THETA(I)*P(I)*LOG(U(I+1))*(1.-U(I)**THETA(I))
          A2=P(I+1)*LOG(U(I))*U(I)**THETA(I)
          A=A1/A2
          THETA(I+1)=1.
       37 THET=THETA(I+1)-(U(I+1)**THETA(I+1))*(THETA(I+1)+A)
          THTA1=1.-U(I+1)**THETA(I+1)
          THTA2=U(I+1)**THETA(I+1)*LOG(U(I+1))*(THETA(I+1)+A)
          THTA=THTA1-THTA2
          D=THET/THTA
          IF(ABS(D)-.01) 36,36,35
       35 THETA(I+1)=THETA(I+1)-D
          GO TO 37
       36 X(I+1)=X(I)-P(I+1)*THETA(I+1)**2.
       13 CONTINUE
          IF(X(N)) 16,19,15
       15 IF(M1) 40,40,19
       16 THETA(1)=THETA(1)-.1
      M1=1
      GO TO 39
   19 DO 71 I=1,N
   71 PUNCH 108,I,R(I),P(I)
      PUNCH 100
      PUNCH 109
      PUNCH 100
      DO 61 I=1,N
      ITHTA=THETA(I)
      DUMMY=ITHTA
      DIFF=THETA(I)-DUMMY
      IF(DIFF-.5) 52,53,53
   52 IRED(I)=DUMMY-1.
      GO TO 54
   53 IRED(I)=DUMMY
   54 IF(THETA(I)) 56,56,51
   56 THETA(I)=1.
   51 REDNT=THETA(I)-1.
      PUNCH 101,I,REDNT,IRED(I)
      RED=IRED(I)
      PR=PR+P(I)*(RED+1.)**2.
   61 REL=REL*(1.-(1.-R(I))**(RED+1.))
      PR=PATT-PR
      PUNCH 100
      PUNCH 103,PR,REL
      END
```

SAMPLE DATA CARDS ****

```
 N  PATT
 3  50.

 R     P
.85    5.
.95    8.
.75    2.
```

TABLE D3. COMPUTER OUTPUT FOR THREE STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  3
CONSTRAINT LIMIT = 50.00

STAGE NO      RELIABILITY           P(I)
    1            .8500              5.00
    2            .9500              8.00
    3            .7500              2.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.=  .9000   USE    1
STAGE   2   NO OF REDUNDANT ELE.=  .3160   USE    0
STAGE   3   NO OF REDUNDANT ELE.= 1.7603   USE    2

SLACK IN CONSTRAINT =   4.0000   SYS.REL.=   .9141
```

TABLE D4. COMPUTER OUTPUT FOR EIGHT STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  8
CONSTRAINT LIMIT =300.00

STAGE NO       RELIABILITY            P(I)
     1            .8500               5.00
     2            .9500               8.00
     3            .7500               2.00
     4            .5500               1.00
     5            .6000               7.00
     6            .6500               3.00
     7            .7000               4.00
     8            .8000               5.00

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.3000   USE    1
STAGE   2   NO OF REDUNDANT ELE.=  .5748   USE    1
STAGE   3   NO OF REDUNDANT ELE.= 2.3109   USE    2
STAGE   4   NO OF REDUNDANT ELE.= 4.3310   USE    4
STAGE   5   NO OF REDUNDANT ELE.= 2.2511   USE    2
STAGE   6   NO OF REDUNDANT ELE.= 2.6317   USE    3
STAGE   7   NO OF REDUNDANT ELE.= 2.1578   USE    2
STAGE   8   NO OF REDUNDANT ELE.= 1.5341   USE    2

SLACK IN CONSTRAINT =   13.0001   SYS.REL.=   .8384
```

Appendix E. Computer Flow Chart, Symbol Table, Computer Program and Results for Example 5.

Fig. B1. COMPUTER FLOW CHART

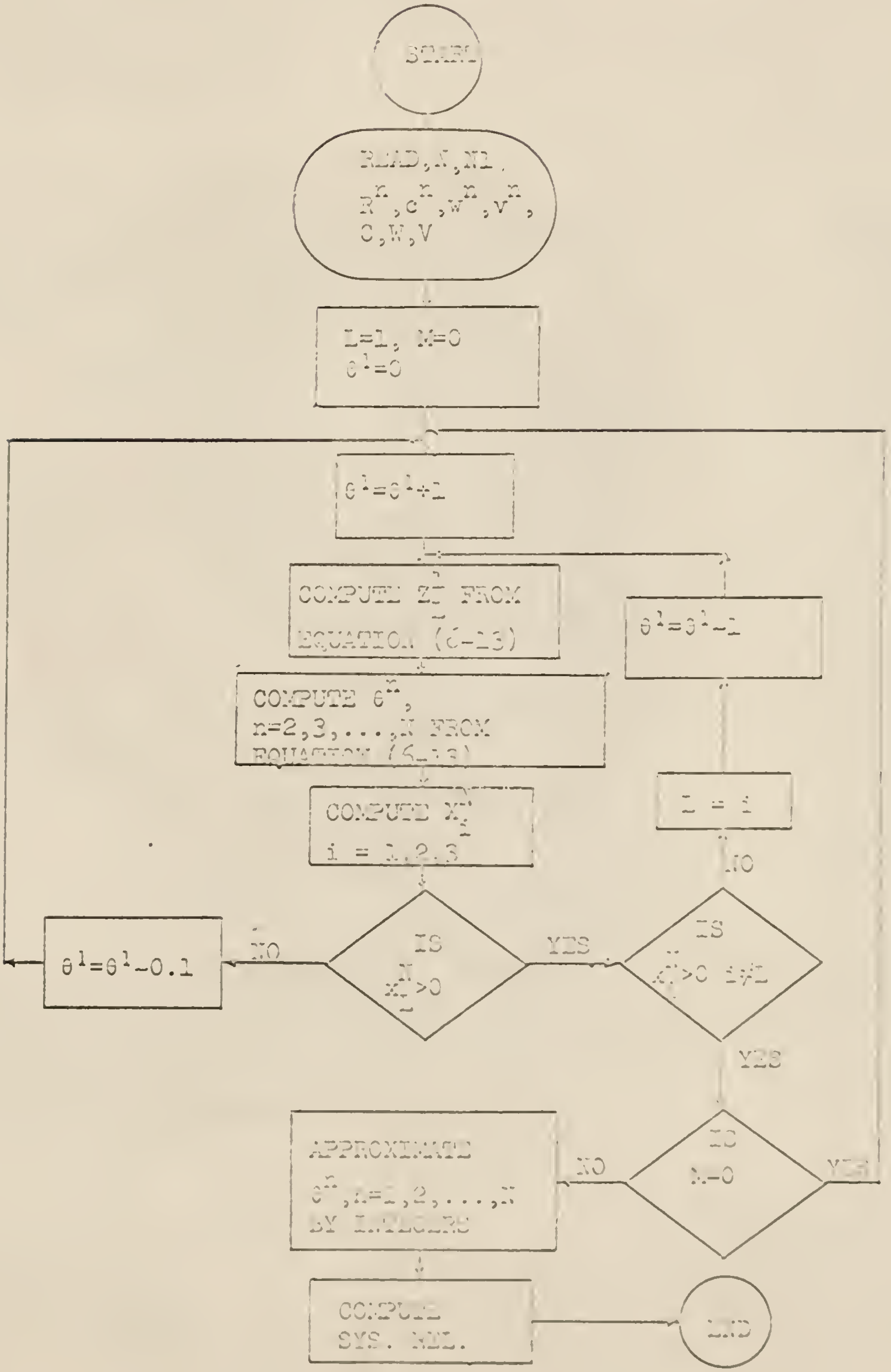
START
READ, N, N1, R^n, c^n, w^n, v^n, C, W, V
L=1, M=0
$\theta^1=0$
$\theta^1=\theta^1+1$
COMPUTE Z_i^1 FROM EQUATION (6-13)
$\theta^1=\theta^1-1$
COMPUTE θ^n, n=2,3,...,N FROM EQUATION (6-13)
COMPUTE X_i^N i = 1,2,3
L = i
NO
IS $x_i^N>0$
NO
$\theta^1=\theta^1-0.1$
YES
IS $x_i^N>0$ $i\neq L$
YES
IS M=0
NO
YES
APPROXIMATE θ^n, n=1,2,...,N BY INTEGERS
COMPUTE SYS. REL.
END

Table E1. Symbol table for computer program.

Program Symbol	Mathematical Symbol	Explanation
REL	R_s	Total system reliability
THETA(I)	θ^n	Number of elements at nth stage
N1		Number of constraints
PATT(J)		Constraint limit for jth resource
C(I,J)		Coefficients in ith stage and jth constraint
X(I,J)	$x_j^n (j=1,2,\ldots N1)$	jth resource left over allocating i stages
UR(I)	$1 - R^n$	Unreliability per element at nth stage
L		Constraint considered to be binding
ITHTA		Integer portion of θ^n
IRED(I)	$\theta^n - 1$	Number of redundant elements at nth stage

TABLE E2. COMPUTER PTOGRAM FOR EXAMPLE 5.

```
C   C  MAXIMIZATION OF SYSTEM RELIABILITY SUBJECT TO MULTIPLE LINEAR CONST.
       DIMENSION THETA(10),PATT(10),R(10),C(10,10),K(10),UR(10),X(10,10)
       DIMENSION IRED(10)
   100 FORMAT(1H )
   101 FORMAT(41HOPTIMUM LIES AT THE INTERSECTION OF CONST,I3,3HAND,I3)
   102 FORMAT(28HDATA FOR THE PROBLEM FOLLOWS)
   103 FORMAT(13HNO OF STAGES=I3)
   104 FORMAT(8HSTAGE NO,5X11HRELIABILITY)
   105 FORMAT(3XI3,9XF6.4)
   106 FORMAT(28HOPTIMUM POLICY IS AS FOLLOWS)
   107 FORMAT(15X20HSYSTEM RELIABILITY =,2XF6.4)
   108 FORMAT(6HSTAGE ,I3,3X21HNO OF REDUNDANT ELE.=,F7.4,3X,3HUSE,I5)
     1 FORMAT(2I5)
     3 FORMAT(F8.2)
     5 FORMAT(F6.4)
     7 FORMAT(F10.2)
       L=1
       REL=1.
       M=0
       THETA(1)=0
       READ 1,N,N1
       DO 23I=1,N1
    23 K(I)=0
       K(1)=L
       DO 2 J=1,N1
     2 READ 3,PATT(J)
       DO 6 I=1,N
       READ 5,R(I)
       DO 6J=1,N1
     6 READ 7,C(I,J)
       PUNCH 102
       PUNCH 103,N
       PUNCH 100
       PUNCH 104
     8 THETA(1)=THETA(1)+1.
    18 K2=0
       DO 9 J=1,N1
     9 X(1,J)=PATT(J)-C(1,J)*(THETA(1))
       UR(1)=1.-R(1)
       Z1=UR(1)**THETA(1)*LOG(UR(1))
       Z2=C(1,L)*(1.-UR(1)**THETA(1))
       Z=Z1/Z2
       DO 11 I=2,N
       UR(I)=1.-R(I)
       THT1=(Z*C(I,L))/(Z*C(I,L)+LOG(UR(I)))
       THETA(I)=LOG(THT1)/LOG(UR(I))
       DO 11 J=1,N1
   X(I,J)=X(I-1,J)-C(I,J)*THETA(I)
11 CONTINUE
   IF(X(N,L)) 14,19,19
19 DO 12 I=1,N1
   IF(X(N,I)) 13,12,12
13 IF(I-K(I)) 16,17,16
16 K(I)=I
   L=I
   M=0
   THETA(1)=THETA(1)-1.
   GO TO 18
17 K2=I
12 CONTINUE
   IF(K2) 24,24,25
24 IF(M) 8,8,20
14 THETA(1)=THETA(1)-.1
   M=1
   GO TO 18
20 DO 71 I=1,N
71 PUNCH 105, I,R(I)
   PUNCH 100
   PUNCH 106
   PUNCH 100
   DO 61 I=1,N
   ITHTA=THETA(I)
   DUMMY=ITHTA
   DIFF=THETA(I)-DUMMY
   IF(DIFF-.5) 52,53,53
52 IRED(I)=DUMMY-1.
   GO TO 54
53 IRED(I)=DUMMY
54 IF(THETA(I)) 56,56,51
56 THETA(I)=1.
51 REDNT=THETA(I)-1.
   PUNCH 108,I,REDNT,IRED(I)
   RED=IRED(I)
61 REL=REL*(1.-(1.-R(I))**(RED+1.))
   PUNCH 100
   PUNCH 107,REL
   GO TO 30
25 PUNCH 101,K2,L
30 STOP
   END
   SAMPLE DATA CARDS   *****


    N     N1
    5     3
50.            FIRST CONSTRAINT LIMIT
132.           SECOND CONSTRAINT LIMIT
155.           THIRD CONSTRAINT LIMIT

  DATA STAGE WISE
.90
3.
5.
8.
.75
2.
4.
9.
.65
4.
9.
6.
.80
1.
7.
7.
.85
2.
7.
8.
```

TABLE E3. COMPUTER OUTPUT FOR FIVE STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  5

STAGE NO       RELIABILITY
     1            .9000
     2            .7500
     3            .6500
     4            .8000
     5            .8500

OPTIMUM POLICY IS AS FOLLOWS

STAGE    1    NO OF REDUNDANT ELE.= 1.9000    USE     2
STAGE    2    NO OF REDUNDANT ELE.= 3.6120    USE     4
STAGE    3    NO OF REDUNDANT ELE.= 4.0561    USE     4
STAGE    4    NO OF REDUNDANT ELE.= 2.7182    USE     3
STAGE    5    NO OF REDUNDANT ELE.= 2.2408    USE     2

                    SYSTEM RELIABILITY =    .9878
```

TABLE E4. COMPUTER OUTPUT FOR TEN STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES= 10

STAGE NO      RELIABILITY
     1            .7500
     2            .6000
     3            .9500
     4            .8000
     5            .8500
     6            .9000
     7            .8000
     8            .5500
     9            .7000
    10            .6500

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.8000   USE    2
STAGE   2   NO OF REDUNDANT ELE.= 2.6784   USE    3
STAGE   3   NO OF REDUNDANT ELE.=  .4220   USE    0
STAGE   4   NO OF REDUNDANT ELE.= 1.6214   USE    2
STAGE   5   NO OF REDUNDANT ELE.= 1.2735   USE    1
STAGE   6   NO OF REDUNDANT ELE.=  .9286   USE    1
STAGE   7   NO OF REDUNDANT ELE.= 1.3740   USE    1
STAGE   8   NO OF REDUNDANT ELE.= 3.7186   USE    4
STAGE   9   NO OF REDUNDANT ELE.= 2.0632   USE    2
STAGE  10   NO OF REDUNDANT ELE.= 2.3859   USE    2

                SYSTEM RELIABILITY =   .7676
```

Appendix F. Computer Flow Chart, Symbol Table, Computer Program and Results for Example 6.

Fig. F1. COMPUTER FLOW CHART

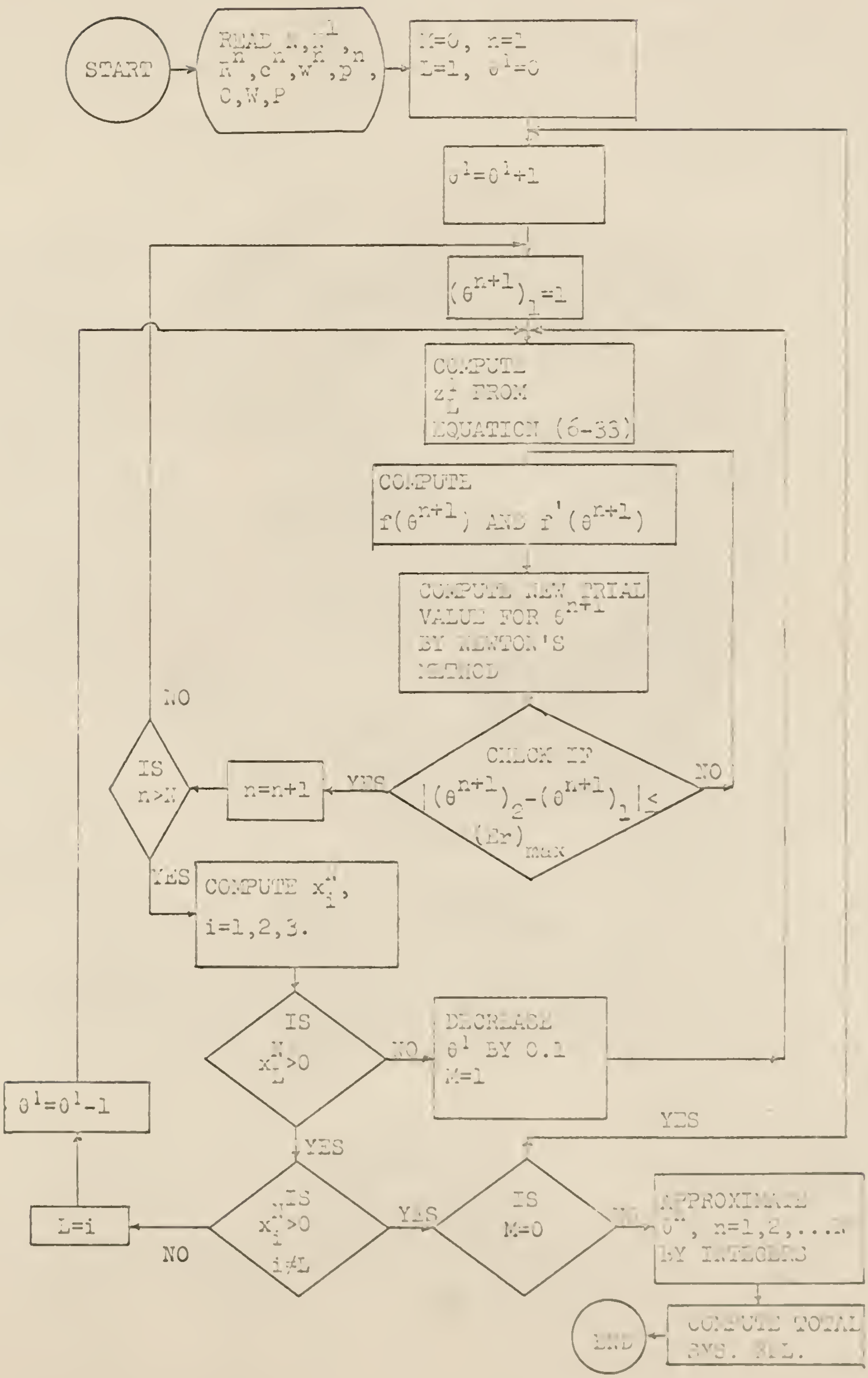
START
READ N, θ1, K n, c n, w n, p n, C, W, P
M=0, n=1
L=1, θ1=0
θ1=θ1+1
(θn+1)1=1
COMPUTE zL± FROM EQUATION (6-33)
COMPUTE f(θn+1) AND f'(θn+1)
COMPUTE NEW TRIAL VALUE FOR θn+1 BY NEWTON'S METHOD
CHECK IF |(θn+1)2-(θn+1)1| ≤ (Er)max
NO
YES
n=n+1
IS n>N
NO
YES
COMPUTE xiN, i=1,2,3.
IS xLN>0
NO
DECREASE θ1 BY 0.1 M=1
YES
θ1=θ1-1
L=i
NO
IS xiN>0 i≠L
YES
IS M=0
YES
NO
APPROXIMATE θn, n=1,2,...N BY INTEGERS
COMPUTE TOTAL SYS. REL.
END

Table F1. Symbol table for computer program.

Program Symbol	Mathematical Symbol	Explanation
REL	R_s	Total system reliability.
PR		Total of first constraint consumed.
CST		Total of second (cost) constraint consumed.
WT		Total of third (weight) constraint consumed.
THETA(I)	θ^n	Number of elements at nth stage.
N1		Number of constraints.
R(I)	R^n	Reliability per unit at nth stage.
U(I)	$1 - R^n$	Unreliability at nth stage per element.
P(I)	p^n	Coefficient at nth stage in first constraint.
C(I)	c^n	Coefficient at nth stage in cost constraint.
W(I)	w^n	Coefficient at nth stage in weight constraint.
X(I,J)	x_j^n	jth resource left over after allocating first n stages.
Z	z^n	Adjoint variable at nth stage.
THET	$f(\theta^n)$	Value of function given by equation (4-35) or (4-38) or (4-41).
THTA	$f'(\theta^n)$	Derivatives of above functions.
L		Constraint to be active.
ITHTA		Integer portion of θ^n.
IRED(I)	$1 - R^n$	Number of redundant elements at nth stage.

TABLE F2   COMPUTER PROGRAM FOR EXAMPLE 6

```
C   C  OPTIMIZATION OF SYS REL SUBJECT TO NON LINEAR MULTIPLE CONSTRAINTS
       DIMENSION R(10),P(10),W(10),C(10),X(10,10),THETA(10),U(10)
       DIMENSION IRED(10),K(10)
   100 FORMAT(1H )
   101 FORMAT(2I5,3F10.2)
   102 FORMAT(F10.5,3F5.2)
   103 FORMAT(28HDATA FOR THE PROBLEM FOLLOWS)
   104 FORMAT(13HNO OF STAGES=I3)
   105 FORMAT(8HSTAGE NO,5X11HRELIABILITY)
   106 FORMAT(3XI3,9XF6.4)
   107 FORMAT(28HOPTIMUM POLICY IS AS FOLLOWS)
   108 FORMAT(6HSTAGE ,I3,3X21HNO OF REDUNDANT ELE.=,F7.4,3X,3HUSE,I5)
   109 FORMAT(15X20HSYSTEM RELIABILITY =,2XF6.4)
   110 FORMAT(41HOPTIMUM LIES AT THE INTERSECTION OF CONST,I3,3HAND,I3)
   111 FORMAT(9HTOTAL PR=F6.2,9HTOTAL WT=F6.2,10HTOTAL CST=F6.2)
       L=1
       REL=1.
       PR=0
       WT=0
       CST=0
       M=0
       THETA(1)=0
       READ 101,N,N1,PATT1,PATT2,PATT3
       DO 1000 I=1,N1
  1000 K(I)=0
       K(1)=L
       DO 1 I=1,N
     1 READ 102,R(I),P(I),C(I),W(I)
       PUNCH 103
       PUNCH 104,N
       PUNCH 100
       PUNCH 105
     2 THETA(1)=THETA(1)+1.
     3 K2=0
       X(1,1)=PATT1-P(1)*THETA(1)**2.
       X(1,2)=PATT2-C(1)*(THETA(1)+EXP(THETA(1)/4.))
       X(1,3)=PATT3-W(1)*THETA(1)*EXP(THETA(1)/4.)
       U(1)=1.-R(1)
       DO 11 I=2,N
       U(I)=1.-R(I)
       THETA(I)=1.
       F=U(I-1)**THETA(I-1)*LOG(U(I-1))/(1.-U(I-1)**THETA(I-1))
     4 GO TO (5,6,7),L
     5 Z=F/(2.*P(I-1)*THETA(I-1))
       A=LOG(U(I;))/(2.*Z*P(I))
       THET=THETA(I)-U(I)**THETA(I)*(THETA(I)+A)
       THTA=1.-U(I)**THETA(I)-(THETA(I)+A)*U(I)**THETA(I)*LOG(U(I))
       GO TO 8
```

```
 6 Z=F/(C(I-1)*(1.+.25*EXP(THETA(I-1)/4.)))
   A=LOG(U(I))/(Z*C(I))
   B=1.+.25*EXP(THETA(I)/4.)
   THET=B-U(I)**THETA(I)*(B+A)
   THT1=(1.-U(I)**THETA(I))*EXP(THETA(I)/4.)/16.
   THT2=U(I)**THETA(I)*LOG(U(I))*(B+A)
   THTA=THT1-THT2
   GO TO 8
 7 Z=F/(W(I-1)*EXP(THETA(I-1)/4.)*(1.+.25*THETA(I-1)))
   A=LOG(U(I))/(Z*W(I))
   B=EXP(THETA(I)/4.)*(1.+.25*THETA(I))
   THET=B-U(I)**THETA(I)*(B+A)
   THT1=.25*EXP(THETA(I)/4.)+B/4.-U(I)**THETA(I)*LOG(U(I))*(B+A)
   THT2=U(I)**THETA(I)*(.25*EXP(THETA(I)/4.)+B/4.)
   THTA=THT1-THT2
 8 D=THET/THTA
   IF(ABS(D)-.01) 10,10,9
 9 THETA(I)=THETA(I)-D
   GO TO 4
10 X(I,1)=X(I-1,1)-P(I)*THETA(I)**2.
   X(I,2)=X(I-1,2)-C(I)*(THETA(I)+EXP(THETA(I)/4.))
   X(I,3)=X(I-1,3)-W(I)*THETA(I)*EXP(THETA(I)/4.)
11 CONTINUE
   IF(X(N,L)) 18,12,12
12 DO 16 I=1,N1
   IF(X(N,I)) 13,16,16
13 IF(I-K(I)) 14,15,14
14 K(I)=I
   L=I
   M=0
   THETA(1)=THETA(1)-1.
   GO TO 3
15 K2=I
16 CONTINUE
   IF(K2) 17,17,27
17 IF(M) 2,2,19
18 THETA(1)=THETA(1)-.1
   M=1
   GO TO 3
19 DO 20 I=1,N
20 PUNCH 106,I,R(I)
   PUNCH 100
   PUNCH 107
   PUNCH 100
   DO 26 I=1,N
   ITHTA=THETA(I)
   DUMMY=ITHTA
   DIFF=THETA(I)-DUMMY
   IF(DIFF-.5) 21,22,22
21 IRED(I)=DUMMY-1.
```

```
   GO TO 23
22 IRED(I)=DUMMY
23 IF(THETA(I)) 24,24,25
24 THETA(I)=1.
25 REDNT=THETA(I)-1.
   PUNCH 108,I,REDNT,IRED(I)
   RED=IRED(I)
   REL=REL*(1.-(1.-R(I))**(RED+1.))
   PR=PR+P(I)*(RED+1.)**2.
   WT=WT+W(I)*(RED+1.)*EXP((RED+1.)/4.)
26 CST=CST+C(I)*((RED+1.)+EXP((RED+1.)/4.))
   PUNCH 100
   PUNCH 109,REL
   PUNCH 111,PR,WT,CST
   GO TO 28
27 PUNCH 110,K2,L
28 STOP
   END
```

SAMPLE DATA CARDS *****

```
  N    N1    PATT1     PATT2     PATT3
  5     3    110.      175.      200.

  R       P     C     W
.80      1.    7.    7.
.90      3.    5.    8.
.65      4.    9.    6.
.75      2.    4.    9.
.85      2.    7.    8.
```

TABLE F3. COMPUTER OUTPUT FOR THREE STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=  3

STAGE NO      RELIABILITY
     1            .8000
     2            .9000
     3            .6500

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.4000   USE     1
STAGE   2   NO OF REDUNDANT ELE.=  .8666   USE     1
STAGE   3   NO OF REDUNDANT ELE.= 2.1465   USE     2

                SYSTEM RELIABILITY =    .9097
TOTAL PR= 52.00TOTAL WT= 87.57TOTAL CST= 89.84
```

TABLE F4. COMPUTER OUTPUT FOR FIVE STAGE PROBLEM

```
DATA FOR THE PROBLEM FOLLOWS
NO OF STAGES=   5

STAGE NO      RELIABILITY
     1            .8000
     2            .8500
     3            .9000
     4            .7500
     5            .6500

OPTIMUM POLICY IS AS FOLLOWS

STAGE   1   NO OF REDUNDANT ELE.= 1.6000   USE    2
STAGE   2   NO OF REDUNDANT ELE.= 1.2816   USE    1
STAGE   3   NO OF REDUNDANT ELE.= 1.0075   USE    1
STAGE   4   NO OF REDUNDANT ELE.= 1.6882   USE    2
STAGE   5   NO OF REDUNDANT ELE.= 2.3981   USE    2

                SYSTEM RELIABILITY =   .9045
TOTAL PR= 83.00TOTAL WT=192.48TOTAL CST=146.12
```

OPTIMIZATION OF SYSTEMS RELIABILITY BY THE DISCRETE MAXIMUM PRINCIPLE

by

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B.E. (Electrical Engineering), University of Bombay
Bombay, India, 1964

B.E. (Mechanical Engineering), University of Bombay
Bombay, India, 1965

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSIT
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1967

ABSTRACT

The objective of this report is to obtain solutions to reliability optimization problems subject to single and multiple, linear and nonlinear constraints. The optimization technique employed is a discrete version of the maximum principle.

In a complex system where a number of stages are connected in series, each stage consisting of a number of elements in parallel, it is possible to raise the reliability of the system to one by employing an infinitely large number of elements at each stage. However, increasing the number of elements at each stage increases the cost, weight and volume of the system. Since resources such as these are limited, there arises a systems reliability optimization problem. The two types of this problem considered in this report are as follows.

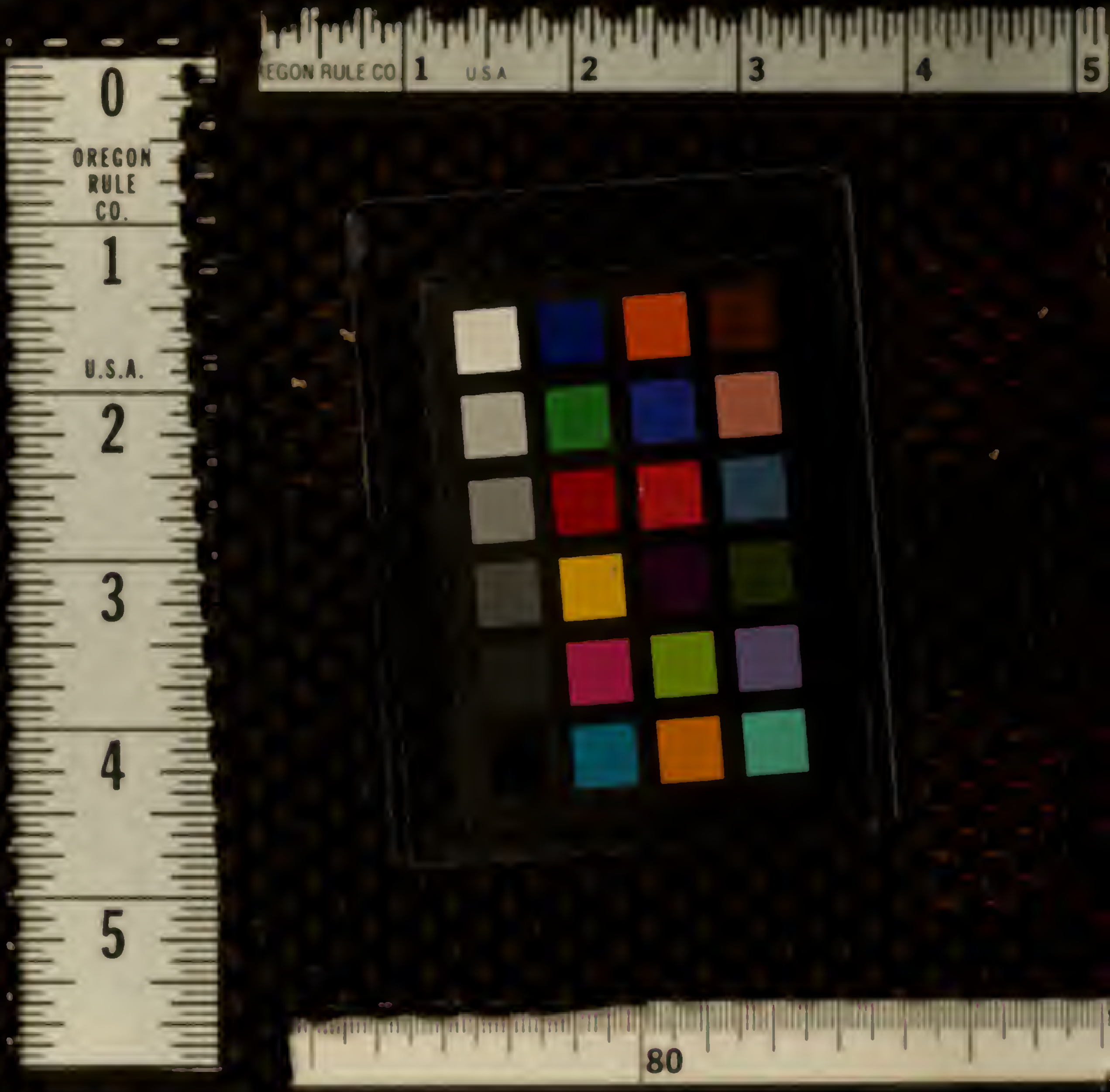
(1) Maximization of the system reliability subject to (a) a single linear or nonlinear constraint, (b) multiple linear or nonlinear constraints.

(2) Minimization of cost while achieving a certain minimum level of reliability.

Examples 1,2,4,5 and 6 are of the type 1. In Examples 1 and 2 a single linear constraint is considered. Example 2 is an extension of example 1, where the maximum number of elements that can be used at any one stage is restricted to some value. Example 4 considers a single nonlinear constraint. Examples 5 and 6 deal with multiple linear and nonlinear constraints respectively. Example 3 is of type 2 where a linear cost function is minimized while achieving a minimum level of reliability.

In all these problems a simple computational procedure is developed based on the discrete version of the maximum principle. Computer programs

for an IBM 1620 are given for these problems. The numerical method developed and the computer programs are for an arbitrary number of stages. Two numerical examples are solved in each problem.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OREGON RULE CO.
1
U.S.A.
2
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EGON RULE CO
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